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Graph Plant Capacity Notions in a Non-parametric Framework: From Input- and Output- to Non-Oriented Plant Capacity

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Abstract

Output-oriented plant capacity in a non-parametric framework is a concept that has been rather widely applied since about twenty-five years. Conversely, input-oriented plant capacity in this framework is a notion of more recent date. In this contribution, we unify the building blocks needed for determining both plant capacity measures and define new graph or non-oriented plant capacity concepts. We empirically illustrate the differences between these various plant capacity notions using a secondary data set. This shows the viability of these new definitions for the applied researcher.

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1 Introduction

The concept of plant capacity has been introduced in the economic literature by Johansen (1968). Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) provide an operational way to measure this concept using a non-parametric frontier framework focusing on a single output and multiple outputs, respectively. Plant capacity utilisation can then be determined from data on observed inputs and outputs by computing a pair of output-oriented efficiency measures relative to a general specification of a non-parametric frontier technology. This has led to a series of empirical applications mainly in fisheries (e.g., Felthoven (2002)) and in the health care sector (for instance, Karagiannis (2015)). There have also occurred some methodological refinements. One example is the inclusion of this plant capacity notion in a decomposition of the Malmquist productivity index (see De Borger and Kerstens (2000)). More recently, Cesaroni, Kerstens, and Van de Woestyne (2017) use the same non-parametric frontier framework to define a new input-oriented measure of plant capacity utilisation based on a couple of input-oriented efficiency measures.

In this methodological contribution, we want to propose a new step and show how new graph or non-oriented plant capacity concepts naturally follow from rewriting the existing output- and input-oriented plant capacity utilisation notions. It is also the first time these graph plant capacity notions are empirically applied. These new plant capacity concepts are more general than the existing ones and provide new tools for the applied researcher.

This contribution is structured as follows. Section 2 provides some basic definitions related to the technology and its representation. The next Section 3 summarizes the existing output- and input-oriented plant capacity utilisation notions and reports the similarities in the building blocks needed for these plant capacity notions. In Section 4 we propose the new graph or non-oriented plant capacity notions based on some existing graph or non-oriented efficiency measures. We also establish some relations between these different plant capacity notions. Section 5 develops a simple numerical example to illustrate the existing and new plant capacity notions within the simplest possible setting. Section 6 offers and empirical application using a secondary data set. The final section concludes.

2 Technology: Basic Definitions

This section introduces some basic notation and defines the production technology. Given an N-dimensional input vector $x \in \mathbb{R}^N_+$ and an M-dimensional output vector $y \in \mathbb{R}^M_+$, the production possibility set or production technology T is defined as follows: $T = \{(x, y) \mid x \text{ can produce at least y}\}$. Associated with technology T, the input set denotes all input vectors x capable of producing at least a given output vector y: $L(y) = \{x \mid (x, y) \in T\}$. Analogously, the output set associated with T denotes all output vectors y that can be produced from at most a given input vector x: $P(x) = \{y \mid (x, y) \in T\}$.

In this contribution, we assume that the production technology T satisfies some combination of the following standard assumptions:

(T.1) Possibility of inaction and no free lunch, i.e., $(0, 0) \in T$ and if $(0, y) \in T$, then $y = 0$.

(T.2) T is a closed subset of $\mathbb{R}^N_+ \times \mathbb{R}^M_+$.

(T.3) Strong input and output disposal, i.e., if $(x, y) \in T$ and $(x', y') \in \mathbb{R}^N_+ \times \mathbb{R}^M_+$, then $(x', -y') \geq (x, -y) \Rightarrow (x', y') \in T$.

(T.4) T is convex.

Briefly commenting these traditional assumptions on the production technology, it is useful to recall the following (see, e.g., Hackman (2008) for details). Inaction is feasible, and there is no free lunch. Technology is closed. We assume strong or free disposability of inputs and outputs in that inputs can be wasted and outputs can be discarded at no opportunity costs. Finally, technology is convex. In our empirical analysis not all these axioms are simultaneously maintained.[1]

The radial input efficiency measure characterizes the input set $L(y)$ completely. It can be defined as follows:

$$DF_i(x, y) = \min\{\theta \mid \theta \geq 0, \theta x \in L(y)\} = \min\{\theta \mid \theta \geq 0, (\theta x, y) \in T\}. \quad (1)$$

This radial input efficiency measure has the main property that it is smaller than or equal to unity ($DF_i(x, y) \leq 1$), with efficient production on the boundary (isoquant) of $L(y)$ represented by unity. Furthermore, the radial input efficiency measure has a cost interpretation (see, e.g., Hackman (2008)).

[1] For example, note that the convex variable returns to scale technology need not satisfy inaction.

The radial output efficiency measure offers a complete characterization of the output set $P(x)$ and can be defined as follows:

$$DF_o(x,y) = \max\{\varphi \mid \varphi \geq 0, \varphi y \in P(x)\} = \max\{\varphi \mid \varphi \geq 0, (x, \varphi y) \in T\}. \tag{2}$$

Its main properties are that it is larger than or equal to unity ($DF_o(x,y) \geq 1$), with efficient production on the boundary (isoquant) of the output set $P(x)$ represented by unity. In addition, this radial output efficiency measure has a revenue interpretation (e.g., Hackman (2008)).

In the short run, we can partition the input vector x into a fixed (x^f) and variable part (x^v). In particular, we denote $x = (x^f, x^v)$ with $x^f \in \mathbb{R}_+^{N_f}$ and $x^v \in \mathbb{R}_+^{N_v}$ such that $N = N_f + N_v$. Fixed inputs are impossible to adjust in the short run, while variable inputs are under complete control of management.

Similar to Färe, Grosskopf, and Valdmanis (1989), a short-run technology $T^f = \{(x^f, y) \in \mathbb{R}_+^{N_f} \times \mathbb{R}_+^M \mid$ there exist some x^v such that (x^f, x^v) can produce at least $y\}$ and the corresponding input set $L^f(y) = \{x^f \in \mathbb{R}_+^{N_f} \mid (x^f, y) \in T^f\}$ and output set $P^f(x^f) = \{y \mid (x^f, y) \in T^f\}$ can be defined. Note that technology T^f is in fact obtained by a projection of technology $T \in \mathbb{R}_+^{N+M}$ into the subspace $\mathbb{R}_+^{N_f+M}$ (i.e., by setting all variable inputs equal to zero). By analogy, the same applies to the input set $L^f(y)$ and the output set $P^f(x^f)$.

Denoting the radial output efficiency measure of the short-run output set $P^f(x^f)$ by $DF_o^f(x^f, y)$, this short-run output-oriented efficiency measure can be defined as follows:

$$DF_o^f(x^f, y) = \max\{\varphi \mid \varphi \geq 0, \varphi y \in P^f(x^f)\} = \max\{\varphi \mid \varphi \geq 0, (x^f, \varphi y) \in T^f\}. \tag{3}$$

The sub-vector input efficiency measure reducing only the variable inputs is defined as follows:

$$DF_i^{SR}(x^f, x^v, y) = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v) \in L(y)\} = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v, y) \in T\}. \tag{4}$$

Next, we need the following particular definition: $L(0) = \{x \mid (x, 0) \in T\}$ is the input set with zero output level. The sub-vector input efficiency measure reducing variable inputs evaluated relative to this input set with a zero output level is as follows:

$$DF_i^{SR}(x^f, x^v, 0) = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v) \in L(0)\} = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v, 0) \in T\}. \tag{5}$$

For the applications in Sections 5 and 6 respectively, we assume a convex non-parametric frontier technology under the flexible or variable returns to scale assumption (VRS). Given data on K observations ($k = 1, \ldots, K$) consisting of a vector of inputs and outputs $(x_k, y_k) \in \mathbb{R}^N_+ \times \mathbb{R}^M_+$, this technology can algebraically be represented by

$$T^{VRS} = \left\{ (x, y) \mid x \geq \sum_{k=1}^{K} z_k x_k, y \leq \sum_{k=1}^{K} z_k y_k, \sum_{k=1}^{K} z_k = 1 \text{ and } z_k \geq 0 \right\}. \qquad (6)$$

The activity vector z of real numbers summing to unity represents the convexity axiom. The convex technology satisfies axioms (T.1) (except inaction) to (T.4).

Commonly, it is assumed that the input and output data satisfy a series of conditions (Färe, Grosskopf, and Lovell (1994, p. 44-45)): (i) each producer employs non-negative amounts of each input to produce non-negative amounts of each output; (ii) there is an aggregate production of positive amounts of every output as well as an aggregate utilization of positive amounts of every input; and (iii) each producer employs a positive amount of at least one input to produce a positive amount of at least one output.

3 Plant Capacity Concepts: Basic Definitions

Recall the informal definition of plant capacity by Johansen (1968, p. 362) as "the maximum amount that can be produced per unit of time with existing plant and equipment, provided that the availability of variable factors of production is not restricted." This clearly output-oriented plant capacity notion has been admirably made operational by Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) using a pair of output-oriented efficiency measures. We now recall the definition of this output-oriented plant capacity utilization.

Definition 3.1. The output-oriented plant capacity utilization PCU_o is defined as follows:

$$PCU_o(x, x^f, y) = \frac{DF_o(x, y)}{DF_o^f(x^f, y)},$$

where $DF_o(x, y)$ and $DF_o^f(x^f, y)$ are output efficiency measures including, respectively excluding, the variable inputs as defined before in (2) and (3).

Since $1 \leq DF_o(x, y) \leq DF_o^f(x^f, y)$, notice that $0 < PCU_o(x, x^f, y) \leq 1$. Thus, output-oriented plant capacity utilization has an upper limit of unity. Following the terminology introduced by Färe, Grosskopf, and Kokkelenberg (1989), one can distinguish between a so-called biased plant capacity measure $DF_o^f(x^f, y)$ and an unbiased plant capacity measure $PCU_o(x, x^f, y)$ depending on whether the measure ignores inefficiency or adjusts for the eventual existence of inefficiency. Taking the ratio of efficiency measures eliminates any existing inefficiency and yields in this sense a cleaned concept of output-oriented plant capacity.

Recently, Kerstens, Sadeghi, and Van de Woestyne (2018) have argued and empirically illustrated that the output-oriented plant capacity utilization $PCU_o(x, x^f, y)$ may be unrealistic in that the amounts of variable inputs needed to reach the maximum capacity outputs may simply be unavailable at either the firm or the industry level. This is linked to what Johansen (1968) called the attainability issue. Hence, Kerstens, Sadeghi, and Van de Woestyne (2018) define a new attainable output-oriented plant capacity utilization at the firm level as follows:

Definition 3.2. An attainable output-oriented plant capacity utilization $APCU_o$ at level $\bar{\lambda} \in \mathbb{R}_+$ is defined by

$$APCU_o(x, x^f, y, \bar{\lambda}) = \frac{DF_o(x, y)}{ADF_o^f(x^f, y, \bar{\lambda})},$$

where the attainable output-oriented efficiency measure ADF_o^f at a certain level $\bar{\lambda} \in \mathbb{R}_+$ is defined by

$$\begin{aligned} ADF_o^f(x^f, y, \bar{\lambda}) &= \max\{\varphi \mid \varphi \geq 0, 0 \leq \theta \leq \bar{\lambda}, \varphi y \in P(x^f, \theta x^v)\} \\ &= \max\{\varphi \mid \varphi \geq 0, 0 \leq \theta \leq \bar{\lambda}, (x^f, \theta x^v, \varphi y) \in T\}. \end{aligned} \quad (7)$$

Again, since $1 \leq DF_o(x, y) \leq ADF_o^f(x^f, y, \bar{\lambda})$, notice that $0 < APCU_o(x, x^f, y, \bar{\lambda}) \leq 1$.

Cesaroni, Kerstens, and Van de Woestyne (2017) define a new input-oriented plant capacity measure using a pair of input-oriented efficiency measures.

Definition 3.3. The input-oriented plant capacity utilization (PCU_i) is defined as follows:

$$PCU_i(x, x^f, y) = \frac{DF_i^{SR}(x^f, x^v, y)}{DF_i^{SR}(x^f, x^v, 0)},$$

where $DF_i^{SR}(x^f, x^v, y)$ and $DF_i^{SR}(x^f, x^v, 0)$ are the sub-vector input efficiency measures defined in (4) and (5), respectively.

Since $0 < DF_i^{SR}(x^f, x^v, 0) \leq DF_i^{SR}(x^f, x^v, y)$, notice that $PCU_i(x, x^f, y) \geq 1$. Thus, input-oriented plant capacity utilization has a lower limit of unity. Similar to the previous case, one can distinguish between a so-called biased plant capacity measure $DF_i^{SR}(x^f, x^v, 0)$ and an unbiased plant capacity measure $PCU_i^{SR}(x, x^f, y)$, the latter being cleaned of any prevailing inefficiency.

The following proposition presents a first new result. It shows that the building blocks needed for calculating the amount of $PCU_o(x, x^f, y)$ and $PCU_i(x, x^f, y)$ can be expressed in similar models (i.e., maximization for output-orientation and minimization for input-orientation) using the same constraints but with different objective functions and different bounds on the decision variables.

Proposition 3.1.

i) The short-run output-oriented radial technical efficiency measure $DF_o^f(x^f, y)$ is equivalently solved as follows:

$$DF_o^f(x^f, y) = \max\{\varphi \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}, \tag{8}$$

whereby $\theta \geq 0$ allows to expand the observed variable inputs.

ii) The input-oriented short-run efficiency measure reducing variable inputs evaluated relative to the input set with a zero output level ($DF_i^{SR}(x^f, x^v, 0)$) is equivalently solved as follows:

$$DF_i^{SR}(x^f, x^v, 0) = \min\{\theta \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}, \tag{9}$$

whereby $\varphi \geq 0$ allows for an adjustment of the observed outputs.

iii) The output-oriented technical efficiency measure $DF_o(x, y)$ is equivalently solved as follows:

$$DF_o(x, y) = \max\{\varphi \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}, \tag{10}$$

whereby $\theta \leq 1$ allows to contract the observed variable inputs.

iv) The input efficiency measure reducing only the variable inputs ($DF_i^{SR}(x^f, x^v, y)$) is equivalently solved as follows:

$$DF_i^{SR}(x^f, x^v, y) = \min\{\theta \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}, \tag{11}$$

whereby $\varphi \geq 1$ allows for an adjustment of the observed outputs.

Proof. See Appendix A. □

We can make the following remarks regarding this first new result. Although these remarks are more general by nature, it might be useful checking out the corresponding models in Appendix B assuming the convex non-parametric technology T^{VRS} (see (6)). First, it is important to understand that in these new formulations, expressions (8) to (11) all use the same constraints (i.e., $(x^f, \theta x^v, \varphi y) \in T$). In the cases of output-orientation, maximization is needed while input-orientation requires minimization. Also notice the difference in the objective functions (i.e., φ in the case of output-orientation and θ for input-orientation). In particular, model (8) aims to maximize the outputs by releasing the variable inputs, while model (9) aims to minimize the variable inputs by releasing the outputs. The same result holds true for models (10) and (11).

Second, notice that models (8) and (10) are identical except for the bounds applied to the decision variables θ and φ. For $DF_o^f(x^f, y)$, we have $\theta \leq 1$ and $\varphi \geq 1$ that prevent to increase the inputs and decrease the output components, while for $DF_o(x, y)$, we have $\theta \geq 0$ and $\varphi \geq 0$. The same result holds true for models (9) and (11).

Third, note that in the new formulation of $DF_i^{SR}(x^f, x^v, 0)$ (i.e., model (9)), the right-hand side of the output constraints is not zero. In fact, this model aims to obtain the minimum amount of variable inputs such that the outputs are not restricted.

Since the input- and output-oriented plant capacity utilisation concepts share the same structure, we are now in a position to extend these notions to the full space of inputs and outputs.

4 Graph Efficiency Measurement and Plant Capacity Utilisation

4.1 New Developments

Methodological research on efficiency (or inefficiency) measurement has early on focused on measurement in the full space of inputs and outputs, which has been referred to as "graph efficiency" measurement in the seminal book by Färe, Grosskopf, and Lovell (1985). An extensive survey of such graph or non-oriented efficiency measures is provided by Russell and Schworm (2011).[2]

[2] A survey of similar input-oriented efficiency measures can be found in the earlier article of Russell and Schworm (2009).

Färe, Grosskopf, and Lovell (1985, p. 110-111) define the hyperbolic efficiency index by

$$E_H(x, y) = \max\{\theta \mid (\theta^{-1}x, \theta y) \in T\}, \tag{12}$$

which can be considered as the first formulation of an efficiency index in the full input and output space. This index contracts inputs and expands outputs along a (particular) hyperbolic path to the frontier and maps into the $[1, \infty)$ interval. This hyperbolic graph efficiency measure (12) extends the analysis of the radial input- and output-oriented efficiency measures by allowing for the adjustment of both inputs and outputs in the measurement of efficiency. However, this hyperbolic graph efficiency measure is rather restrictive since it constrains the search for more efficient production plans to a hyperbolic path along which all inputs are reduced and all outputs are increased in the same proportion.[3]

Under constant returns to scale (CRS), Färe, Grosskopf, and Zaim (2002) show that this hyperbolic efficiency index is equal to both $DF_i(x, y)^{\frac{1}{2}}$ and $DF_o(x, y)^{-\frac{1}{2}}$ (i.e., the conventional output- and input-oriented distance functions that can be solved using standard linear programming (LP) techniques). But, under VRS, the hyperbolic efficiency index may not be obtained by solving an LP-problem. To linearise this problem, Färe, Grosskopf, Lovell, and Pasurka (1989) introduce a linear approximation of the non-linear set of constraints. However, Zofío and Lovell (2001) and Zofío and Prieto (2001) show that this approximation is only acceptable close to the efficient frontier. Hence, when a unit becomes more inefficient the approximation worsens. To resolve the linearisation problem of the hyperbolic distance function under VRS, Färe, Margaritis, Rouse, and Roshdi (2016) propose an LP-based computational algorithm for estimating the exact value of the hyperbolic graph efficiency measure by connecting it to the directional distance function proposed by Chambers, Chung, and Färe (1998). We refer to Section 4.2 for more information concerning this distance function.

Starting from Färe, Grosskopf, and Lovell (1985, p. 126) one can define the generalized Farrell graph measure as follows:

$$E_{FGL}(x, y) = \max\{\frac{\varphi + \theta}{2} \mid \theta \geq 1, \varphi \geq 1, (\theta^{-1}x, \varphi y) \in T\}. \tag{13}$$

This generalization of the hyperbolic efficiency measure permits the proportional reduction in all inputs to differ from the proportional increase in all outputs when searching for a more efficient production plan.[4]

[3] Färe, Grosskopf, and Zaim (2002) show that the hyperbolic graph efficiency measure can be given a ratio-based return to the dollar interpretation.

[4] Färe, Grosskopf, and Lovell (1985, p. 126) do not consider the constraints $\theta \geq 1, \varphi \geq 1$. But, it can

A practical difficulty with this measure is that it must be computed from a non-linear programming problem whose solution is not easily obtained. Therefore, we propose the alternative graph efficiency measure

$$E_G(x,y) = \max\{\frac{\varphi}{\theta} \mid \theta \leq 1, \varphi \geq 1, (\theta x, \varphi y) \in T\}, \tag{14}$$

which, although closely related, avoids this difficulty. Instead of combining input and output radial measures in an additive way (see expression (13)), we define the graph efficiency measure as the ratio between these input and output component measures.

Contrary to $E_{FGL}(x,y)$, $E_G(x,y)$ can more easily be computed since it requires solving an ordinary linear fractional programming problem which can be achieved using linear programming (see Appendix B).[5]

Using the same structure, the sub-vector graph efficiency measure $E_G^f(x^f, x^v, y)$ is defined by

$$E_G^f(x^f, x^v, y) = \max\{\frac{\varphi}{\theta} \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{15}$$

It simultaneously reduces all variable inputs and expands all outputs.

The sub-vector graph efficiency measure $E_G^{SR}(x^f, x^v, y)$ defined by

$$E_G^{SR}(x^f, x^v, y) = \max\{\frac{\varphi}{\theta} \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}, \tag{16}$$

gives complete freedom to adjust both the variable inputs as well as the outputs. Notice that models (15) and (16) are identical except for the bounds on the decision variables φ and θ.

Both efficiency measures $E_G^f(x^f, x^v, y)$ and $E_G^{SR}(x^f, x^v, y)$ aim to maximize the ratio of changes in outputs over changes in variable inputs. But, the main difference between these efficiency measures is as follows. In the efficiency measure $E_G^f(x^f, x^v, y)$, the output components increase and the variable inputs decrease, while in the short-run efficiency measure $E_G^{SR}(x^f, x^v, y)$ both variable inputs and output components are allowed to adjust in a flexible way.

Note that if the variable φ is ignored in the objective function of models (15) and (16) determining $E_G^f(x^f, x^v, y)$ and $E_G^{SR}(x^f, x^v, y)$, we then obtain the reciprocal of the efficiency

be shown that it is necessary to include these constraints to avoid that $E_{FGL}(x,y) < 1$ for some strongly efficient units.

[5]Note that Pastor, Ruiz, and Sirvent (1999) proceeded in a similar way to transform the non-linear part of the non-radial Russell graph measure proposed by Färe, Grosskopf, and Lovell (1985, p. 154). Thereafter, Tone (2001) extended this proposal into the so-called slack-based measure (SBM).

measures $DF_i^{SR}(x^f, x^v, y)$ and $DF_i^{SR}(x^f, x^v, 0)$, determined by models (11) and (9), respectively. Similarly, if the variable θ is ignored in the objective function of models (15) and (16), we then obtain the efficiency measures $DF_o(x, y)$ and $DF_o^f(x^f, x^v, y)$, determined by models (10) and (8), respectively.

We now introduce different graph plant capacity notions using the above defined input-oriented, output-oriented and graph efficiency measures.

Definition 4.1. The graph non-oriented plant capacity utilization $GPCU$ is defined as follows:

$$GPCU(x, x^f, y) = \frac{E_G^f(x^f, x^v, y)}{E_G^{SR}(x^f, x^v, y)}.$$

Note that $GPCU(x, x^f, y) \leq 1$ since $0 < E_G^f(x^f, x^v, y) \leq E_G^{SR}(x^f, x^v, y)$.

We have by definition no limitations on the available amounts of variable inputs for the original output-oriented plant capacity utilisation $PCU_o(x, x^f, y)$. However, in some empirical settings this is not realistic and we have to limit the amount of variable inputs available at either the firm or the industry level (see Kerstens, Sadeghi, and Van de Woestyne (2018) for details). One way to limit the amounts of variable inputs is as follows. It may be reasonable that the amount of increase in the variable inputs is proportional to the increase in the amount of outputs. Hence, we define an output-oriented plant capacity utilisation in graph space by considering the changes of inputs. The graph output-oriented plant capacity utilization can now be defined as follows:

Definition 4.2. The graph output-oriented plant capacity utilization $GPCU_o$ is defined as follows:

$$GPCU_o(x, x^f, y) = \frac{DF_o(x, y)}{GDF_o^{SR}(x^f, x^v, y)},$$

where $GDF_o^{SR}(x^f, x^v, y)$ is the optimal value of φ in model (16).

Note that $GPCU_o(x, x^f, y) \leq 1$ just like $GPCU(x, x^f, y) \leq 1$ and $PCU_o(x, x^f, y) \leq 1$.

As already mentioned in Section 3, the original output-oriented plant capacity utilization PCU_o suffers from the attainability issue. Hence, Kerstens, Sadeghi, and Van de Woestyne (2018) introduce $APCU_o$ (see Definition 3.2) by imposing bounds on the availability of its variable inputs in a general way. Definition 4.2 can also provide another approach to solve the attainability issue. In fact, if we ignore the variable θ in the objective function of model (16), then $GPCU_o(x, x^f, y)$ becomes $PCU_o(x, y)$. Actually, the variable θ in the objective function prevents an increase of the variable inputs and allows the variable inputs to increase only as far as the ratio of changes in outputs over changes of variable inputs is maximized. However, in the following proposition, we show that $GPCU_o$ is a special case of $APCU_o$.

Proposition 4.1. *Assume that θ^* is the optimal value of θ in model (16). Then, we have:*

i) $GDF_o^{SR}(x^f, x^v, y) = ADF_o^f(x^f, y, \theta^*)$;

ii) $GPCU_o(x, x^f, y) = APCU_o(x, x^f, y, \theta^*)$.

Proof. See Appendix A. □

Proposition 4.1 shows that if we use the attainable level $\bar{\lambda} = \theta^*$, then the graph output-oriented and the attainable output-oriented plant capacity utilization concepts offer the same results. Note that $APCU_o(x, x^f, y, \bar{\lambda})$ depends on the attainable level $\bar{\lambda}$ that is determined by the decision maker. But, in some situations he or she may not be able to determine this attainability level. Therefore, the defined attainability in Definition 4.2 can offer a good choice to determine some reasonable bound on available variable inputs.

In a similar way, the graph input-oriented plant capacity utilization can be defined as:

Definition 4.3. The graph input-oriented plant capacity utilization ($GPCU_i$) is defined as follows:

$$GPCU_i(x, x^f, y) = \frac{DF_i(x, x^f, y)}{GDF_i^{SR}(x^f, x^v, y)},$$

where $GDF_i^{SR}(x^f, x^v, y)$ is the optimal value of θ in model (16).

Note that $GPCU_i(x, x^f, y) \geq 1$ just like $PCU_i(x, x^f, y) \geq 1$.

Proposition 4.2. *The following relations between the original graph as well as the special case graph output- and graph input-oriented plant capacity utilization notions as well as their components can be established:*

i) $DF_i^{SR}(x^f, x^v, 0) \leq GDF_i^{SR}(x^f, x^v, y) \leq GDF_o^{SR}(x^f, x^v, y) \leq DF_o^f(x^f, y)$;

ii) $PCU_o(x, x^f, y) \leq GPCU_o(x^v, x^f, y)$;

iii) $GPCU_i(x^v, x^f, y) \leq PCU_i(x^v, x^f, y)$.

iv) $GPCU(x, x^f, y) \gtreqless GPCU_o(x, x^f, y) \gtreqless GPCU_i(x, x^f, y)$

v) $GPCU(x, x^f, y) \leq PCU_i(x^v, x^f, y)$

vi) $GPCU(x, x^f, y) \gtreqless PCU_o(x, x^f, y)$ *and* $GPCU(x, x^f, y)^{-1} \geq PCU_o(x, x^f, y)$.

Proof. See Appendix A. □

Note that $GPCU(x, x^f, y)$ is in general different from its special input- and output-oriented graph versions $GPCU_i(x, x^f, y)$ and $GPCU_o(x, x^f, y)$, respectively. Furthermore, $GPCU(x, x^f, y)$ is also distinct from the traditional input- and output-oriented plant capacity concepts $PCU_i(x^v, x^f, y)$ and $PCU_o(x, x^f, y)$, respectively. Finally, $GPCU_i(x, x^f, y)$ and $GPCU_o(x, x^f, y)$ differ in general from $PCU_i(x^v, x^f, y)$ and $PCU_o(x, x^f, y)$, respectively.

4.2 Link to Graph Capacity Measure Based on Directional Distance Function

The directional distance function $E_D(x, y, g_x, g_y)$ proposed by Chambers, Chung, and Färe (1998) is defined by

$$E_D(x, y, g_x, g_y) = \max\{\beta \mid (x - \beta g_x, y + \beta g_y) \in T\}. \quad (17)$$

This distance function simultaneously seeks to expand outputs and contract inputs in the direction of the vector $(-g_x, g_y) \in \mathbb{R}^N_- \times \mathbb{R}^M_+$. The latter directional vector determines how the input-output vector (x, y) is projected onto the boundary of T at $(x - \beta^* g_x, y + \beta^* g_y)$, whereby $\beta^* = E_D(x, y, g_x, g_y)$.[6]

Partitioning the input vector x into fixed x^f and variable x^v, the sub-vector directional distance function is defined by

$$E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = \max\{\beta \mid (x^f, x^v - \beta g_x^v, y + \beta g_y) \in T\}. \quad (18)$$

Färe and Grosskopf (2000) are the first to use this directional distance function as a tool for defining a theoretical graph-oriented plant capacity utilization indicator.[7] In fact, they expand the outputs and contract the fixed inputs in the graph-oriented plant capacity measure using the directional distance function.

[6]See Russell and Schworm (2011) for an almost complete overview of graph or non-oriented efficiency measures, including the directional distance function.

[7]Yang and Fukuyama (2018) seemingly independently also define a graph-oriented plant capacity measure using the directional distance function while also distinguishing between good and bad outputs. These same authors are -to the best of our knowledge- the first to empirically apply this particular graph-oriented plant capacity utilization indicator based on the directional distance function.

We now show in the following propositions that the building blocks needed for computing all plant all capacity measures hitherto defined in this contribution can be obtained from the directional distance functions E_D and E_D^{SR} by choosing appropriate direction vectors. For the proofs, we refer to Appendix A.

Proposition 4.3. *Assume that (θ^*, φ^*) is an optimal solution obtained by solving the following model:*

$$DF_o^f(x^f, y) = \max\{\varphi \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}. \tag{19}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{DF_o^f(x^f, y)} x^v, \quad and$$

$$g_y = \frac{(\varphi^* - 1)}{DF_o^f(x^f, y)} y,$$

then $DF_o^f(x^f, y) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1, 1)$, then we put $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Proposition 4.4. *Assume that (θ^*, φ^*) is an optimal solution obtained by solving the following model:*

$$DF_o(x, y) = \max\{\varphi \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{20}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{DF_o(x, y)} x^v, \quad and$$

$$g_y = \frac{(\varphi^* - 1)}{DF_o(x, y)} y,$$

then $DF_o(x, y) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1, 1)$, then we put $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Proposition 4.5. *Assume that (θ^*, φ^*) is an optimal solution obtained by solving the following model:*

$$DF_i^{SR}(x^f, x^v, 0) = \min\{\theta \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}. \tag{21}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{DF_i^{SR}(x^f, x^v, 0)} x^v, \quad and$$

$$g_y = \frac{(\varphi^* - 1)}{DF_i^{SR}(x^f, x^v, 0)} y,$$

then $DF_i^{SR}(x^f, x^v, 0) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1,1)$, *then we put* $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Proposition 4.6. *Assume that* (θ^*, φ^*) *is an optimal solution obtained by solving the following model:*

$$DF_i^{SR}(x^f, x^v, y) = \min\{\theta \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{22}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{DF_i^{SR}(x^f, x^v, y)} x^v, \quad \textit{and}$$

$$g_y = \frac{(\varphi^* - 1)}{DF_i^{SR}(x^f, x^v, y)} y,$$

then $DF_i^{SR}(x^f, x^v, y) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1,1)$, *then we put* $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Proposition 4.7. *Assume that* (θ^*, φ^*) *is an optimal solution obtained by solving the following model:*

$$E_G^f(x^f, x^v, y) = \max\{\frac{\varphi}{\theta} \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{23}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{E_G^f(x^f, x^v, y)} x^v, \quad \textit{and}$$

$$g_y = \frac{(\varphi^* - 1)}{E_G^f(x^f, x^v, y)} y,$$

then $E_G^f(x^f, x^v, y) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1,1)$, *then we put* $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Proposition 4.8. *Assume that* (θ^*, φ^*) *is an optimal solution obtained by solving the following model:*

$$E_G^{SR}(x^f, x^v, y) = \max\{\frac{\varphi}{\theta} \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}. \tag{24}$$

Letting

$$g_x^v = \frac{(1-\theta^*)}{E_G^{SR}(x^f, x^v, y)} x^v, \quad \textit{and}$$

$$g_y = \frac{(\varphi^* - 1)}{E_G^{SR}(x^f, x^v, y)} y,$$

then $E_G^{SR}(x^f, x^v, y) = E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$.
Note that if $(\theta^*, \varphi^*) = (1, 1)$, *then we put* $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) = 1$.

Therefore, one may consider the graph-oriented plant capacity utilization indicator based on the directional distance function as a special case of our graph-oriented plant capacity utilization indices.

5 Numerical Example

We illustrate the ease of implementing these new graph plant capacity definitions introduced in this contribution by using a small example of artificial data. We refer to Appendix B for an overview on how to compute the necessary components of the plant capacity notions assuming a convex non-parametric technology under VRS. Table 1 contains 16 fictitious observations with two inputs generating a single output: one input is variable, the other one is fixed.

Table 1: Numerical Example of 16 Fictitious Observations

Observations	x^v	x^f	y
1	7	6	2
2	3	5	2
3	5	4	3
4	6	3	3
5	7	4	3
6	4	9	4
7	11	3	4
8	5	6	4
9	6	3	4
10	6	7	5
11	5	8	5
12	8	6	5
13	10	5	5
14	6	10	6
15	7	8	6
16	10	7	6

The results of the traditional input- and output plant capacity measures and their components are reported in Table 2. Observe that the observations with a unit value for $PCU_i(.)$

and $PCU_o(.)$ are different in general: solely observations 4, 5 and 7 have a unity plant capacity in both cases. This confirms that both these plant capacity measures evaluate different things.

Table 2: Input- and Output-oriented Plant Capacity Utilization and Components

	$PCU_i(.)$			$PCU_o(.)$		
	$DF_i^{SR}(x^f, x^v, y)$	$DF_i^{SR}(x^f, x^v, 0)$	$PCU_i(.)$	$DF_o(.)$	$DF_o^f(.)$	$PCU_o(.)$
1	0.4286	0.4286	1.0000	2.6250	2.7500	0.9545
2	1.0000	1.0000	1.0000	1.0000	2.5000	0.4000
3	0.9000	0.9000	1.0000	1.1538	1.5000	0.7692
4	1.0000	1.0000	1.0000	1.3333	1.3333	1.0000
5	0.6429	0.6429	1.0000	1.5000	1.5000	1.0000
6	1.0000	0.7500	1.3333	1.0000	1.5000	0.6667
7	0.5455	0.5455	1.0000	1.0000	1.0000	1.0000
8	0.9500	0.6000	1.5833	1.0577	1.3750	0.7692
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.9333	0.5000	1.8667	1.0400	1.2000	0.8667
11	1.0000	0.6000	1.6667	1.0000	1.2000	0.8333
12	0.7750	0.3750	2.0667	1.0750	1.1000	0.9773
13	0.8000	0.3000	2.6667	1.0000	1.0000	1.0000
14	1.0000	0.5000	2.0000	1.0000	1.0000	1.0000
15	1.0000	0.4286	2.3333	1.0000	1.0000	1.0000
16	1.0000	0.3000	3.3333	1.0000	1.0000	1.0000

The results of the new graph plant capacity concept $GPCU(x, x^f, y)$ as well as its special input- and output-oriented graph versions $GPCU_i(x, x^f, y)$ and $GPCU_o(x, x^f, y)$ are reported in Table 3. This same table also contains their respective component efficiency measures. Observe that observations 1, 5 and 14 with a unit value for $GPCU(x, x^f, y)$ have different values for $GPCU_i(x, x^f, y)$ and $GPCU_o(x, x^f, y)$. By contrast, observations 4, 6-7, 9 and 11 are unity for all three graph capacity notions. For the observations which are different from unity, all three graph capacity notions in general differ.

Table 3: Graph Plant Capacity Utilization, Graph Input- and Output-Oriented Plant Capacity Utilization, and Components

	$GPCU(.)$			$GPCU_i(.)$		$GPCU_o(.)$	
	$E_G^f(.)$	$E_G^{SR}(.)$	$GPCU(.)$	$GDF_i^{SR}(.)$ $= \bar{\lambda}$	$GPCU_i(.)$	$GDF_o^{SR}(.)$	$GPCU_o(.)$ $= APCU(., \bar{\lambda})$
1	2.9815	2.9815	1.0000	0.7714	0.5556	2.3000	1.1413
2	1.0000	1.1786	0.8485	1.8667	0.5357	2.2000	0.4545
3	1.1538	1.2069	0.9560	1.1600	0.7759	1.4000	0.8242
4	1.3333	1.3333	1.0000	1.0000	1.0000	1.3333	1.0000
5	1.6897	1.6897	1.0000	0.8286	0.7759	1.4000	1.0714
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	1.8333	1.8333	1.0000	0.5455	1.0000	1.0000	1.0000
8	1.0577	1.0648	0.9933	1.0800	0.8796	1.1500	0.9197
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	1.0714	1.1077	0.9673	0.8667	1.0769	0.9600	1.0833
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
12	1.2903	1.3630	0.9467	0.6750	1.1481	0.9200	1.1685
13	1.2500	1.5714	0.7955	0.5600	1.4286	0.8800	1.1364
14	1.0000	1.0000	1.0000	0.6667	1.5000	0.6667	1.5000
15	1.0000	1.1667	0.8571	0.7143	1.4000	0.8333	1.2000
16	1.0000	1.5385	0.6500	0.5200	1.9231	0.8000	1.2500

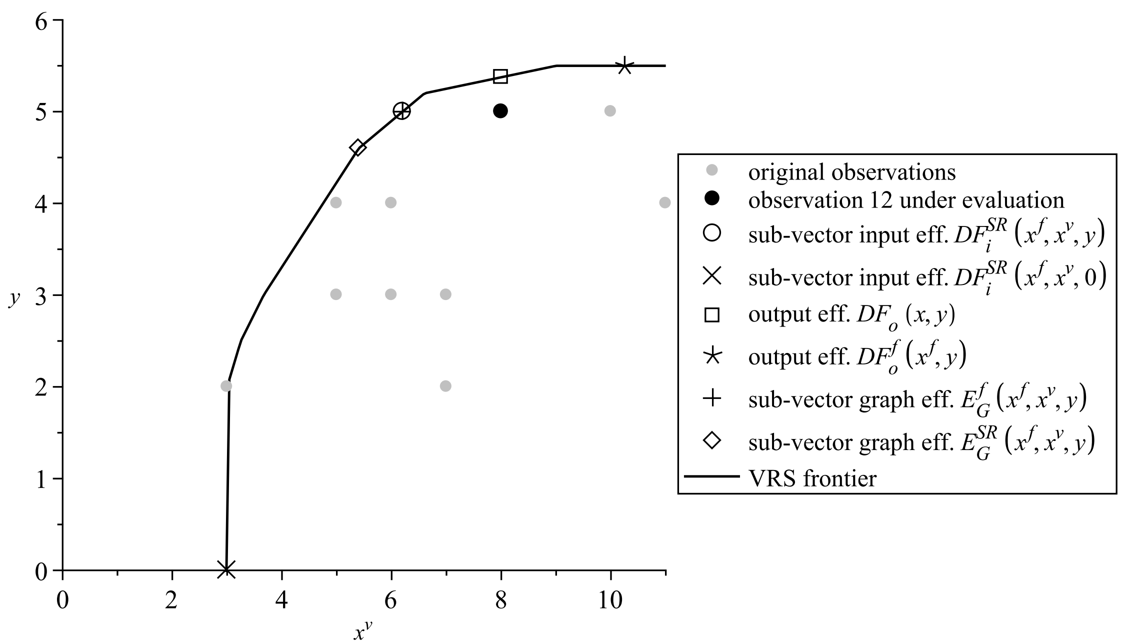

Figure 1: Visualization of the different components of PCU_i, PCU_o and $GPCU$ applied to observation 12

Figure 1 shows the (x^v, y)-combinations determined by the optimal solutions of those models needed for computing PCU_i, PCU_o and $GPCU$ applied to observation 12 (black solid circle). The values mentioned below can be found in Tables 2 and 3 except for the values of φ^* and θ^*. Observe in Figure 1 the original observations (gray solid circles) with fixed input smaller than or equal to the fixed input of observation 12 and the corresponding frontier. Applying (11) to compute $DF_i^{SR}(x^f, x^v, y)$ on this observation results in the efficiency value 0.7750 with $\varphi^* = 1$ and $\theta^* = 0.7750$. Consequently, the variable input is reduced without expanding the output. The resulting (x^v, y)-combination is depicted by the black circle (○) located on the frontier. Applying (9) to compute $DF_i^{SR}(x^f, x^v, 0)$ yields 0.3750 with $\varphi^* = 0$ and $\theta^* = 0.3750$. So, by allowing to reduce outputs to the level of zero, the variable input can be reduced further. Combining the optimal variable input and output level realizes the point visualized by a diagonal cross ($\times$). Since there is only one variable input and one output in this example, $PCU_i(x, x^f, y)$ boils down to the ratio of the horizontal distance from the point ○ to the y-axis over the horizontal distance from the point $\times$ to the y-axis.

Using (10) to compute $DF_o(x, y)$ for observation 12 leads to the optimal value 1.0750 with $\varphi^* = 1.0750$ and $\theta^* = 1$. Now, the output is maximally increased without reducing the variable input. The corresponding optimal (x^v, y)-combination is visualized by the black box ($\square$) in Figure 1. Using (8) to determine $DF_o^f(x^f, y)$ results in the optimal value 1.1000 with $\varphi^* = 1.1000$ and $\theta^* = 1.2813$. So, by allowing an increase of the variable input, the output can be increased by this factor 1.1000. The corresponding optimal (x^v, y)-combination is visualized by the asterisk ($\star$). Since there is only one variable input and one output in this example, $PCU_o(x, x^f, y)$ corresponds with the ratio of the vertical distance from the point $\square$ to the x^v-axis over the vertical distance from the point $\star$ to the x^v-axis.

Computing the sub-vector graph efficiency $E_G^f(x^f, x^v, y)$ defined by (15) for observation 12 leads to the optimal value 1.2903 with $\varphi^* = 1$ and $\theta^* = 0.7750$. Obviously, the maximal ratio $\frac{\varphi^*}{\theta^*}$ is realized by reducing the variable input and keeping the output at the same level. This results in the optimal (x^v, y)-combination depicted in Figure 1 by the cross ($+$). Note that this point coincides for observation 12 with the optimal point realized by DF_i^{SR} and visualized by ○. Using (16), the sub-vector graph efficiency $E_G^{SR}(x^f, x^v, y)$ for observation yields the optimal ratio $\frac{\varphi^*}{\theta^*} = 1.3630$ with $\varphi^* = 0.9200$ and $\theta^* = 0.6750$. Thus, the optimal ratio is obtained by reducing both the variable input and the output. The corresponding optimal (x^v, y)-combination is visualized by the diamond ($\diamond$). The ratio of the ratios $\frac{\varphi^*}{\theta^*}$mentioned above yields $GPCU(x, x^f, y) = 0.9467$.

We explore the differences between these graph plant capacity notions in more detail in the empirical illustration in the next Section 6.

6 Empirical Illustration

6.1 Secondary Data Set

For an overview of the models used in this empirical illustration assuming a convex non-parametric technology under VRS, we refer to Appendix B. We select a secondary data set from the *Journal of Applied Econometrics* Data Archive[8] and opt for an unbalanced panel of three years of French fruit producers from Ivaldi, Ladoux, Ossard, and Simioni (1996) based on annual accounting data collected in a survey. These farms were selected on mainly two criteria: (i) the apple production must be positive, and (ii) the orchard acreage is five acres at least. This short panel covers three successive years from 1984 to 1986. The technology consists of three aggregate inputs producing two aggregate outputs. The three inputs are (i) capital (including land), (ii) labor, and (iii) materials; while the two outputs are (i) the apple production, and (ii) an aggregate of other products. Descriptive statistics for the 405 observations in total and details on the definitions of all variables are available in Appendix 2 of Ivaldi, Ladoux, Ossard, and Simioni (1996). Note that the short length of the panel (just three years) warrants the use of an intertemporal technology that essentially ignores any eventual technical change.

6.2 Empirical Results

The descriptive statistics for the traditional input- and output-oriented plant capacity utilization measures as well as their components are reported in Table 4. We report the average, the standard deviation, and the minima and maxima depending on the context. These descriptive statistics seem to confirm that both concepts clearly differ from one another (see Cesaroni, Kerstens, and Van de Woestyne (2017) for a detailed empirical analysis confirming this observation).

Table 4: Descriptive Statistics of Input- and Output-oriented Plant Capacity Utilization

	$PCU_i(.)$			$PCU_o(.)$		
	$DF_i^{SR}(x^f, x^v, y)$	$DF_i^{SR}(x^f, x^v, 0)$	$PCU_i(.)$	$DF_o(.)$	$DF_o^f(.)$	$PCU_o(.)$
Average	0.5881	0.4233	1.7337	3.4924	5.4149	0.7105
Stand.Dev.	0.1924	0.1950	1.6360	2.6312	4.6781	0.2211
Minimum	0.1868	0.0473	1	1	1	0.0701
Maximum	1	1	21.1414	16.2869	35.2953	1

[8] See the web site: `http://qed.econ.queensu.ca/jae/`.

The descriptive statistics for new graph plant capacity concept $GPCU(x, x^f, y)$ as well as its special input- and output-oriented graph versions $GPCU_i(x, x^f, y)$ and $GPCU_o(x, x^f, y)$ are reported in Table 5. Also the component efficiency measures are listed in the same table. We can make the following observations. First, the descriptive statistics indicate that all three graph capacity notions in general seem to differ.

Table 5: Descriptive Statistics of Various Graph Plant Capacity Utilization Concepts

	$GPCU(.)$			$GPCU_i(.)$		$GPCU_o(.)$	
	$E_G^f(.)$	$E_G^{SR}(.)$	$GPCU(.)$	$\bar{\lambda} = GDF_i^{SR}(.)$	$GPCU_i(.)$	$GDF_o^{SR}(.)$	$GPCU_o(.) = APCU(., \bar{\lambda})$
Average	3.6971	3.9243	0.9430	1.0463	0.6742	4.2375	1.0240
Stand.Dev.	2.7219	2.8763	0.1049	0.4366	0.5713	4.1750	0.4875
Minimum	1	1	0.1521	0.1333	0.1842	0.2804	0.0835
Maximum	16.2869	16.4006	1	2.9549	7.5025	33.1387	3.5658

Second, note that based on Proposition 4.1, $GPCU_o(x, x^f, y) = APCU_o(x, x^f, y, \bar{\lambda})$. Hence, the last column of Table 5 reports the attainable output-oriented plant capacity utilisation at level $\bar{\lambda} = \theta^*$, where θ^* is the optimal value of θ in model (16). Hence, $\bar{\lambda} = GDF_i^{SR}(x, x^f, y)$ which is reported in the fifth column of Table 5. While the attainable output-oriented plant capacity utilization $APCU_o(x, x^f, y, \bar{\lambda})$ depends on the attainable level $\bar{\lambda}$ determined by the decision maker, the defined attainability in Definition 4.2 (i.e., $GPCU_o(x, x^f, y) = APCU_o(x, x^f, y, \bar{\lambda})$) can be a good choice to determine some reasonable bound on the available variable inputs. As can be seen in the fifth column of Table 5, the average of attainability level $\bar{\lambda}$ is 1.0463 with the standard deviation 0.4366. With this proposed attainability level $\bar{\lambda}$ we obtain on average a 1.0240 output magnification, while the maximum increase in outputs amounts to 3.5658 times. It suffices to put this in contrast with the biased plant capacity measure $DF_o^f(x, x^f, y)$. There is a lot of variation in $DF_o^f(x, x^f, y)$ as indicated by the standard deviation and the range is even huge: the maximum increase in outputs amounts to 35.2953 times. Our approach clearly avoids such extreme magnifications and therefore remains attainable.

To test whether any of the above results are statistically significant, we choose the formal test statistic proposed by Li (1996) and refined by Fan and Ullah (1999) and Li, Maasoumi, and Racine (2009) (henceforth Li-test). This Li-test has the null hypothesis that both distributions are equal for a given efficiency score or plant capacity notion. Its alternative hypothesis is simply that both distributions differ. This test is valid for both dependent and independent variables: dependency is a characteristic of frontier estimators, since frontier efficiency depend on sample size, among others. Table 6 reports the Li-test statistics for all plant capacity concepts discussed in this contribution. A glance at Table 6 shows that all plant capacity concepts follow two by two significantly different distributions and thus capture an independent part of reality.

Table 6: Li-Test between All Unbiased Plant Capacity Notions

	$PCU_i(.)$	$PCU_o(.)$	$GPCU(.)$	$GPCU_i(.)$	$GPCU_o(.)$
$PCU_i(.)$		74.606***	143.6798***	124.5037***	17.558***
$PCU_o(.)$	74.606***		71.1327***	21.742***	20.6112***
$GPCU(.)$	143.6798***	71.1327***		145.117***	110.6579***
$GPCU_i(.)$	124.5037***	21.742***	145.117***		55.832***
$GPCU_o(.)$	17.558***	20.6112***	110.6579***	55.832***	

Li test: critical values at 1% level= 2.33($***$); 5% level= 1.64($**$); 10%level= 1.28($*$).

7 Conclusions

While the output-oriented plant capacity concept has been around for about three decades and has been the basis for quite some empirical applications, the input-oriented plant capacity concept is of more recent date. However, this original output-oriented plant capacity utilization suffers from the so-called attainability issue, which has led Kerstens, Sadeghi, and Van de Woestyne (2018) to define an attainable output-oriented plant capacity notion. This contribution has taken a next logical step by looking for graph efficiency measures to define some new graph-oriented plant capacity concepts. Apart from some new definitions, relations between these capacity concepts have been established. A small numerical example has illustrated these various concepts and an empirical application on a secondary data set has revealed the factual differences between these different notions.

We end by outlining a potential avenue for future research. For instance, there are some indications that slacks may play a role in the measurement of plant capacity utilization (e.g., Dupont, Grafton, Kirkley, and Squires (2002), or Vestergaard, Squires, and Kirkley (2003)). Of course, on a priori grounds one would expect that slacks play less role for graph-oriented plant capacity that are more likely to project on the efficient subset of technology than for output- or input-oriented plant capacity notions. But, this issue certainly merits further attention.

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Appendices: Supplementary Material

A Proof of Propositions

Proof of Proposition 3.1:

(i) Suppose that

$$\varphi^* = \max\{\varphi \mid \varphi \geq 0, (x^f, \varphi y) \in T^f\}, \tag{25}$$

and

$$\varphi^{**} = \max\{\varphi \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}. \tag{26}$$

We must show that $\varphi^* = \varphi^{**}$. On the one hand, since $(x^f, \varphi^* y) \in T^f$, then there is a $\bar{\theta} \geq 0$ such that $(x^f, \bar{\theta} x^v, \varphi^* y) \in T$. Hence, $(\bar{\theta}, \varphi^*)$ is a feasible solution of model (26). Thus, $\varphi^{**} \geq \varphi^*$. On the other hand, assume that θ^{**} is the optimal value of variable θ in model (26). So, $(x^f, \theta^{**} x^v, \varphi^{**} y) \in T$. Thus, $(x^f, \varphi^{**} y) \in T^f$. Hence, $\varphi^{**} \leq \varphi^*$. Therefore, we have $\varphi^* = \varphi^{**}$.

(ii) Suppose that

$$\theta^* = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v, 0) \in T\}, \tag{27}$$

and

$$\theta^{**} = \min\{\theta \mid \theta \geq 0, \varphi \geq 0, (x^f, \theta x^v, \varphi y) \in T\}. \tag{28}$$

We must show that $\theta^* = \theta^{**}$. On the one hand, since $(x^f, \theta^* x^v, 0) \in T$, therefore $(\bar{\theta} = \theta^*, \bar{\varphi} = 0)$ is a feasible solution of model (28). Thus, $\theta^{**} \leq \theta^*$. On the other hand, assume that φ^{**} is the optimal value of variable φ in model (28). Hence, $(x^f, \theta^{**} x^v, \varphi^{**} y) \in T$. Based on the strong input disposal assumption (assumption (T.3)), we have $(x^f, \theta^{**} x^v, 0) \in T$. Thus, $\theta^* \leq \theta^{**}$. Therefore, we have $\theta^* = \theta^{**}$.

(iii) Suppose that

$$\varphi^* = \max\{\varphi \mid \varphi \geq 0, (x, \varphi y) \in T\}, \tag{29}$$

and

$$\varphi^{**} = \max\{\varphi \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{30}$$

We must show that $\varphi^* = \varphi^{**}$. Since $\bar{\varphi} = 1$ is a feasible solution of model (29), we have $\varphi^* \geq 1$, hence, $(\theta^* = 1, \varphi^*)$ is a feasible solution of model (30). Thus, $\varphi^{**} \geq \varphi^*$. Also, assume that $(\theta^{**}, \varphi^{**})$ is an optimal solution of model (30). Thus, $(x^f, \theta^{**} x^v, \varphi^{**} y) \in T$. Since $\theta^{**} \leq 1$, hence, $\theta^{**} x^v \leq x^v$. Based on the strong input disposal assumption

(assumption (T.3)), we have $(x^f, x^v, \varphi^{**}y) \in T$. Therefore, $(x, \varphi^{**}y) \in T$. Thus, $\varphi^* \geq \varphi^{**}$. Therefore, we have $\varphi^* = \varphi^{**}$.

(iv) Suppose that

$$\theta^* = \min\{\theta \mid \theta \geq 0, (x^f, \theta x^v, y) \in T\}, \tag{31}$$

and

$$\theta^{**} = \min\{\theta \mid \theta \leq 1, \varphi \geq 1, (x^f, \theta x^v, \varphi y) \in T\}. \tag{32}$$

We must show that $\theta^* = \theta^{**}$. Since $\bar{\theta} = 1$ is a feasible solution of model (31), we have $\theta^* \leq 1$, hence, $(\theta^*, \varphi^* = 1)$ is a feasible solution of model (32). Thus, $\theta^{**} \leq \theta^*$. Also, assume that $(\theta^{**}, \varphi^{**})$ is an optimal solution of model (32). Thus, $(x^f, \theta^{**}x^v, \varphi^{**}y) \in T$. Since $\varphi^{**} \geq 1$, hence, $\varphi^{**}y \geq y$. Based on the strong output disposal assumption (assumption (T.3)), we have $(x^f, \theta^{**}x^v, y) \in T$. Thus, $\theta^* \leq \theta^{**}$. Therefore, we have $\theta^* = \theta^{**}$.

Proof of Proposition 4.1:

(i) Assume that (θ^*, φ^*) and $(\theta^{**}, \varphi^{**})$ are the optimal solutions of determining models $ADF_o^f(x^f, y, \bar{\lambda})$ and $E_G^{SR}(x^f, x^v, y)$, respectively. Note that we assumed that $\bar{\lambda} = \theta^{**}$. On the one hand, since $\theta^* \leq \theta^{**}$, we have $\theta^* x^v \leq \theta^{**}x^v$. Based on the strong output disposal assumption (assumption (T.3)), $(x^f, \theta^{**}x^v, \varphi^* y) \in T$. Thus, (θ^{**}, φ^*) is a feasible solution of determining model $E_G^{SR}(x^f, x^v, y)$, therefore we have $\frac{\varphi^*}{\theta^{**}} \leq \frac{\varphi^{**}}{\theta^{**}}$. Since $\theta^{**} > 0$, thus $\varphi^* \leq \varphi^{**}$. On the other hand, since the optimal solution of determining model $E_G^{SR}(x^f, x^v, y)$ (i.e., $(\theta^{**}, \varphi^{**})$) is also a feasible solution of determining model $ADF_o^f(x^f, y, \bar{\lambda})$, therefore we have $\varphi^{**} \leq \varphi^*$. Therefore, $\varphi^{**} = \varphi^*$.

(ii) Based on part (i), we have that $GDF_o^{SR}(x^f, x^v, y) = ADF_o^f(x^f, y, \bar{\lambda})$. Therefore, $\frac{DF_o(x,y)}{GDF_o^{SR}(x^f,x^v,y)} = \frac{DF_o(x,y)}{ADF_o^f(x^f,y,\bar{\lambda})}$.

Proof of Proposition 4.2:

(i) Proof of $DF_i^{SR}(x^f, x^v, 0) \leq GDF_i^{SR}(x^f, x^v, y)$: Assume that (θ^*, φ^*) is the optimal solutions of determining model $E_G^{SR}(x^f, x^v, y)$. Therefore, (θ^*, φ^*) is also a feasible solution of determining model $DF_i^{SR}(x^f, x^v, 0)$. Hence, we have $DF_i^{SR}(x^f, x^v, 0) \leq \theta^* = GDF_i^{SR}(x^f, x^v, y)$.

Proof of $GDF_i^{SR}(x^f, x^v, y) \leq GDF_o^{SR}(x^f, x^v, y)$: Since $E_G^{SR} = \frac{GDF_o^{SR}(x^f,x^v,y)}{GDF_i^{SR}(x^f,x^v,y)} \geq 1$. Thus, we have $GDF_i^{SR}(x^f, x^v, y) \leq GDF_o^{SR}(x^f, x^v, y)$.

Proof of $GDF_o^{SR}(x^f, x^v, y) \leq DF_o^f(x^f, y)$: Assume that (θ^*, φ^*) is the optimal solutions of determining model $E_G^{SR}(x^f, x^v, y)$. Therefore, (θ^*, φ^*) is also a feasible solution of determining model $DF_o^f(x^f, y)$. Hence, we have $DF_o^f(x^f, y) \geq \varphi^* = GDF_o^{SR}(x^f, x^v, y)$.

(ii) Based on the part (i), we have $DF_o^f(x^f, y) \geq GDF_o^{SR}(x^f, x^v, y)$. Hence, $\frac{DF_o(x,y)}{DF_o^f(x^f,y)} \leq \frac{DF_o(x,y)}{GDF_o^{SR}(x^f,x^v,y)}$. Thus, $PCU_o(x, x^f, y) \leq GPCU_o(x^v, x^f, y)$.

(iii) Based on the part (i), we have $DF_i^{SR}(x^f, x^v, 0) \leq GDF_i^{SR}(x^f, x^v, y)$. Hence, $\frac{DF_i^{SR}(x,x^f,y)}{DF_i^{SR}(x^f,x^v,0)} \geq \frac{DF_i^{SR}(x,x^f,y)}{GDF_i^{SR}(x^f,x^v,y)}$. Thus, $PCU_i(x^v, x^f, y) \geq GPCU_i(x^v, x^f, y)$.

(iv) The results of the numerical example show that $GPCU(x, x^f, y) \underset{<}{\overset{\geq}{=}} GPCU_o(x, x^f, y) \underset{<}{\overset{\geq}{=}} GPCU_i(x, x^f, y)$.

(v) Since $GPCU(x, x^f, y) \leq 1$ and $PCU_i(x^v, x^f, y) \geq 1$, therefore, $GPCU(x, x^f, y) \leq PCU_i(x^v, x^f, y)$.

(vi) The results of the numerical example show that $GPCU(x, x^f, y)$ can be equal, bigger or smaller than $PCU_o(x, x^f, y)$. To prove the other part, since $GPCU(x, x^f, y) \leq 1$ and $PCU_o(x, x^f, y) \leq 1$, therefore, $GPCU(x, x^f, y)^{-1} \geq PCU_o(x, x^f, y)$.

Proof of Proposition 4.3:

Assume that (θ^*, φ^*) is an optimal solution of determining model $DF_o^f(x^f, y)$. Suppose that $\beta^* = DF_o^f(x^f, y)$. We have:

$$x^v - \beta^* g_x^v = x^v - \beta^* \frac{(1-\theta^*)}{DF_o^f(x^f, y)} x^v = \theta^* x^v,$$

and,

$$y + \beta^* g_y = y + \beta^* \frac{(\varphi^* - 1)}{DF_o^f(x^f, y)} y = \varphi^* y.$$

Since $(x^f, \theta^* x^v, \varphi^* y) \in T$, therefore $(x^f, x^v - \beta^* g_x^v, y + \beta^* g_y) \in T$. Hence, $\beta^* = DF_o^f(x^f, y)$ is a feasible solution of determining model $E_D^{SR}(x^v, x^f, y, g_x^v, g_y)$. Thus, $E_D^{SR}(x^v, x^f, y, g_x^v, g_y) \geq DF_o^f(x^f, y)$. Suppose that $\beta^{**} = E_D^{SR}(x^v, x^f, y, g_x^v, g_y) > DF_o^f(x^f, y)$. Therefore,

$$x^v - \beta^{**} g_x^v = x^v - \beta^{**} \frac{(1-\theta^*)}{DF_o^f(x^f, y)} x^v = (1 - \beta^{**}(\frac{(1-\theta^*)}{DF_o^f(x^f, y)}))x^v < \theta^* x^v,$$

and

$$y + \beta^{**} g_y = y + \beta^{**} \frac{(\varphi^* - 1)}{DF_o^f(x^f, y)} y = (1 + \beta^{**}(\frac{(\varphi^* - 1)}{DF_o^f(x^f, y)}))y_o > \varphi^* y.$$

Letting

$$\bar{\theta} = (1 - \beta^{**}(\frac{(1-\theta^*)}{DF_o^f(x^f, y)})).$$

and

$$\bar{\varphi} = (1 + \beta^{**}(\frac{(\varphi^* - 1)}{DF_o^f(x^f, y)})),$$

Since $(x^f, x^v - \beta^{**} g_x^v, y + \beta^{**} g_y) \in T$, we have $(x^f, \bar{\theta} x^v, \bar{\varphi} y) \in T$. Therefore, $(\bar{\theta}, \bar{\varphi})$ is a feasible solution of determining model $DF_o^f(x^f, y)$ such that $\bar{\varphi} > \varphi^*$, which is a contradiction. Thus the proof is complete.

The proofs of Propositions 4.4 to 4.8 all have the same structure as that of Proposition 4.3. To save space, the are omitted.

B Computing Plant Capacity Notions

This appendix presents how the components of various capacity concepts can be estimated in a convex non-parametric frontier framework assuming VRS. To specify the estimation models, we first recall the notations introduced in this contribution. The vector of N inputs ($x \in \mathbb{R}_+^N$) allows producing a vector of M outputs ($y \in \mathbb{R}_+^M$). The vector of inputs x can be partitioned into a fixed (x^f) and variable part (x^v) as $x = (x^f, x^v)$. Assume that for every observed production unit k under observation, ($k = 1, \ldots, K$), both the input (x_k) and corresponding output vectors (y_k) are known. The corresponding fixed and variable input components are denoted by x_k^f and x_k^v, respectively. Finally, since non-parametric frontier technologies are founded on activity analysis, we need a vector of activity variables $z = (z_1, \ldots, z_K)$ indicating the intensity levels at which each of these K observed activities is conducted.

The estimation of the plant and economic capacity components under convexity by using non-parametric frontier methods implies solving the following series of linear programming problems for each observation $(x_o, y_o) \in \{(x_k, y_k) \mid k = 1, \ldots, K\}$.

The output-oriented radial technical efficiency measure $DF_o(x, y)$ defined in (2) is computed by solving the following linear program:

$$
\begin{aligned}
DF_o(x,y) = \quad & \max_{\varphi, z_k} \quad \varphi \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq x_o, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \varphi \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{33}
$$

By decomposing inputs into their fixed and variable components and by using expression (10) in Proposition 3.1, model (33) can be rewritten as follows:

$$
\begin{aligned}
DF_o(x,y) = \quad & \max_{\varphi, \theta, z_k} \quad \varphi \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \leq 1, \varphi \geq 1, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{34}
$$

The short-run output-oriented radial technical efficiency measure $DF_o^f(x^f, y)$ defined in (3) is computed by solving the following linear program:

$$
\begin{aligned}
DF_o^f(x^f,y) = \quad & \max_{\varphi, z_k} \quad \varphi \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \varphi \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{35}
$$

Based on expression (8) in Proposition 3.1, model (35) can be rewritten as follows:

$$
\begin{aligned}
DF_o^f(x^f, y) = \quad & \max_{\varphi,\theta,z_k} \quad \varphi \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 0, \varphi \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{36}
$$

The input-oriented radial technical efficiency measure $DF_i(x, y)$ defined in (1) is computed by solving the following linear program:

$$
\begin{aligned}
DF_i(x, y) = \quad & \min_{\theta,z_k} \quad \theta \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq \theta x_o, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{37}
$$

The input efficiency measure $DF_i^{SR}(x^f, x^v, y)$ defined in (4) is computed by solving the following linear program:

$$
\begin{aligned}
DF_i^{SR}(x^f, x^v, y) = \quad & \min_{\theta,z_k} \quad \theta \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{38}
$$

Based on expression (11) in Proposition 3.1, model (38) can be rewritten as follows:

$$
\begin{aligned}
DF_i^{SR}(x^f, x^v, y) = \quad \min_{\varphi,\theta,z_k} \quad & \theta \\
s.t \quad & \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \sum_{k=1}^{K} z_k = 1, \\
& \theta \leq 1, \varphi \geq 1, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{39}
$$

The input-oriented short-run efficiency measure $DF_i^{SR}(x^f, x^v, 0)$ defined in (5) is computed by solving the following linear program:

$$
\begin{aligned}
DF_i^{SR}(x^f, x^v, 0) = \quad \min_{\theta,z_k} \quad & \theta \\
s.t \quad & \sum_{k=1}^{K} z_k y_k \geq 0, \\
& \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \sum_{k=1}^{K} z_k = 1, \\
& \theta \geq 0, z_k \geq 0, \qquad k = 1, \ldots, K.
\end{aligned}
\tag{40}
$$

Based on expression (9) in Proposition 3.1, model (40) can be rewritten as follows:

$$
\begin{aligned}
DF_i^{SR}(x^f, x^v, 0) = \quad \min_{\varphi,\theta,z_k} \quad & \theta \\
s.t \quad & \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \sum_{k=1}^{K} z_k = 1, \\
& \theta \geq 0, \varphi \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{41}
$$

The attainable output-oriented efficiency measure $ADF_o^f(x^f, y, \bar{\lambda})$ at level $\bar{\lambda} \in \mathbb{R}_+$ defined in (7) is computed by solving the following linear program:

$$
\begin{aligned}
ADF_o^f(x^f, y, \bar{\lambda}) = \quad & \max_{\theta,\varphi,z_k} \quad \varphi \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad 0 \leq \theta \leq \bar{\lambda}, \\
& \qquad z_k \geq 0, \qquad k = 1, \ldots, K.
\end{aligned} \tag{42}
$$

The hyperbolic efficiency measure $E_H(x, y)$ defined in (12) is computed by solving the following non-linear program:

$$
\begin{aligned}
E_H(x, y) = \quad & \max_{\theta,z_k} \quad \theta \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \theta y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq \tfrac{1}{\theta} x_o, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{43}
$$

The generalized Farrell graph measure $E_{FGL}(x, y)$ defined in (13) is computed by solving the following non-linear program:

$$
\begin{aligned}
E_{FGL}(x, y) = \quad & \max_{\theta,\varphi,z_k} \quad \tfrac{\varphi+\theta}{2} \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq \tfrac{1}{\theta} x_o, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 1, \varphi \geq 1, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{44}
$$

The generalized graph measure $E_G(x, y)$ defined in (14) is computed by solving the following non-linear program:

$$
\begin{aligned}
E_G(x,y) = \quad & \max_{\theta,\varphi,z_k} \quad \tfrac{\varphi}{\theta} \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq \theta x_o, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \leq 1, \varphi \geq 1, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{45}
$$

The non-linear program (45) can be transformed into a linear program using the Charnes-Cooper transformation as follows (see Charnes and Cooper (1962)):

$$
\begin{aligned}
E_G(x,y) = \quad & \max_{\theta,\varphi,t,z_k} \quad \varphi \\
& s.t \quad \theta = 1, \\
& \qquad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k \leq \theta x_o, \\
& \qquad \sum_{k=1}^{K} z_k = t, \\
& \qquad \theta \leq t, \varphi \geq t, t \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{46}
$$

The sub-vector graph efficiency measure $E_G^f(x^f, x^v, y)$ defined in (15) is computed by solving the following non-linear program:

$$
\begin{aligned}
E_G^f(x^f, x^v, y) = \quad & \max_{\theta,\varphi,z_k} \quad \tfrac{\varphi}{\theta} \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \leq 1, \varphi \geq 1, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{47}
$$

The sub-vector graph efficiency measure $E_G^{SR}(x^f, x^v, y)$ defined in (16) is computed by solving the following non-linear program:

$$
\begin{aligned}
E_G^{SR}(x^f, x^v, y) = \quad & \max_{\theta, \varphi, z_k} \quad \tfrac{\varphi}{\theta} \\
& s.t \quad \sum_{k=1}^{K} z_k y_k \geq \varphi y_o, \\
& \qquad \sum_{k=1}^{K} z_k x_k^f \leq x_o^f, \\
& \qquad \sum_{k=1}^{K} z_k x_k^v \leq \theta x_o^v, \\
& \qquad \sum_{k=1}^{K} z_k = 1, \\
& \qquad \theta \geq 0, \varphi \geq 0, z_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{48}
$$

Note that the non-linear programming problems (47) and (48) also can be transformed into linear programs using the Charnes-Cooper transformation in a similar way as model (45).