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2015-05

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Physical capital, financial asset and dividend taxation

Ngoc-Sang Pham*

November 30, 2015

Abstract

We consider an infinite-horizon general equilibrium model with heterogeneous agents and financial market imperfections.

First, we prove that, in the case of low productivity, recession takes place at infinitely many dates. When the government levies taxes on asset dividends to finance investments in R&D or human capital, the productivity is fostered, recession is prevented and the economy experiences growth. However, recession may occur if the R&D process is inefficient.

Second, we study the optimal dividend taxation and we show that a high tax rate may promote bubbles in asset prices.

Keywords: Intertemporal equilibrium, recession, growth, R&D, dividend taxation, asset bubbles.

JEL Classifications: C62, D31, D91, G10, E44.

1 Introduction

This paper studies the impact of dividend taxation on economic growth and asset bubbles. To this purpose, we construct a deterministic dynamic general equilibrium model with heterogeneous consumers, a firm and a government. In this model, a long-lived asset is traded and a single good is consumed or/and used to produce. An agent buys the long-lived asset today and may resell it tomorrow after receiving exogenous dividends (in term of consumption good). This asset can be interpreted either as land (Lucas tree) or security,[1] or stock.[2] In the following, it will be referred to as financial asset. Consumers may invest either in physical capital or in financial asset, and borrow by selling a financial asset within the limit of a borrowing constraint: the repayment of each consumer cannot exceed a fraction of her (physical) capital income.

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[1]See Santos and Woodford (1997).

[2]See Kocherlakota (1992).

The representative firm maximizes its profit by computing its capital demand. The government taxes the dividends on the financial asset. The government spends these taxes to finance research and development (henceforth, R&D) activities that improve in turn the firm's productivity.

First, the existence of equilibrium is proved by slightly adapting the method in Le Van and Pham (2015).

Second, we wonder whether recessions arise and how to avoid them with a positive growth. A recession in the productive sector is said to appear if the capital used for production falls below some critical level, say $\bar{k}$. We show that recessions appear at infinitely many dates if the firm's productivity is too low. With a critical threshold $\bar{k} = 0$, we recover the result in Le Van and Pham (2015).

The novelty of our work is that taxation on asset dividends allows the government to avoid recessions and possibly promote economic growth according to the following mechanism: the government levies taxes on consumers' asset dividends and spends these taxes to finance the R&D. The R&D increases the Total Factor Productivity (henceforth, TFP) and rules out the recessions, promoting economic growth in the end. Given a low initial productivity, recession is always prevented if (1) the R&D process is efficient or (2) the dividends are high or (3) the tax on dividends is high. In some cases, the economy may grow without bounds if the R&D process is sufficiently efficient. By contrast, when these three conditions are violated, the economy cannot escape recession.

When the government increases the tax rate on dividends (τ), the net dividends decrease but the production level increases. Hence, the total amount of good may decrease or increase. It is valuable to study the optimal dividend taxation to grasp this trade-off (see Section 4). In this respect, we assume that the government maximizes the aggregate consumption of the economy at the steady state by choosing the tax rate. If the TFP or the efficiency of R&D or the asset dividends are high, the government should choose the highest feasible tax rate on dividends. By contrast, if these factors are low, the government has to apply the lowest tax rate. In the intermediate case for TFP, R&D and dividends, the optimal level of dividend taxation depends on these three factors as follows. It is increasing in the R&D efficiency and the firm's TFP, but decreasing in the dividend. Our analysis contributes to the optimal taxation theory. The main difference is that in the standard literature, (Chamley, 1986; Judd, 1985; Kocherlakota, 2010) study capital and labor income taxations while we focus on dividend taxation. Moreover we consider heterogeneous consumers and financial frictions while Kocherlakota (2010) studies a representative agent without financial friction.[3]

The last part of the paper focuses on the impact of dividend taxation on asset bubbles. We allow for non-stationary taxes. Following Kocherlakota (1992), Santos and Woodford (1997), we say that asset bubbles arise at equilibrium if the fundamental value (the sum of discounted values) of asset dividends (after tax) exceeds the assets' equilibrium price. Given stationary technology and dividends, we show

[3]The representative agent in Kocherlakota (2010) faces a unique intertemporal constraint. We refer to Aiyagari (1995), Bhandari et al (2013) for optimal taxation in models with incomplete markets.

that, if the dividend tax is not so high, there is no asset bubble. However, when the dividend tax tends to 1, there is room for asset bubbles. The intuition is straightforward. The higher the tax on dividends, the lower the asset fundamental value. This value may be lower than the asset equilibrium price when the dividend tax tends to 1.

The paper is organized as follows. Section 2 presents the model and provides some basic equilibrium properties. Section 3 investigates the role of dividend taxation on recessions and economic growth. Section 4 studies the optimal dividend taxation. Section 5 considers the role of dividend taxation on asset bubbles. Section 6 concludes. Formal proofs are gathered in Appendix.

2 The framework

We revisit Le Van and Pham (2015), a deterministic infinite-horizon general equilibrium model, by introducing a government. Time is discrete: $t = 0, \dots, \infty$. There are three types of agents: a representative firm without market power, m heterogeneous households and the government.

Households

Each household invests in physical or financial asset, and consumes.

Consumption good: there is a single good which can be consumed or used to produce. p_t is its price at period t and $c_{i,t}$ the amount of good consumed by agent i.

Physical capital: $\delta \in (0, 1)$ denotes the capital depreciation rate, while r_t the return of capital. If agent i buys $k_{i,t} \geq 0$ units of physical capital at date $t-1$, then she will receive in the following period $(1-\delta)k_{i,t}$ units of physical capital (after depreciation) and returns $r_t k_{i,t}$.

Financial asset: if agent i buys $a_{i,t}$ units of financial asset at a price q_t at date t, she will receive in the following period ξ_{t+1} units of consumption good as dividend. Then, she will resell $a_{i,t}$ units of financial asset at a price q_{t+1}. This asset takes on different meanings: land,[4], security (Santos and Woodford, 1997) or stock (Kocherlakota, 1992).

Differently from the existing literature, we introduce a government taxing the revenue from asset dividends. For each unit of dividend, any consumer must pay τ units of consumption good.

Each household i takes the sequence of prices $(p, q, r) := (p_t, q_t, r_t)_{t=0}^{\infty}$ as given,

[4] When nobody can borrow, i.e., $f_i = 0$ for any i. See constraint (1).

and solves the following program:

$$(P_i(p,q,r)): \max_{(c_{i,t},k_{i,t+1},a_{i,t})_{t=0}^{\infty}} \Big[\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t})\Big] \tag{1}$$

$$\text{subject to}: \quad k_{i,t+1} \geq 0 \tag{2}$$

$$p_t(c_{i,t}+k_{i,t+1}-(1-\delta)k_{i,t})+q_t a_{i,t}$$
$$\leq r_t k_{i,t}+q_t a_{i,t-1}+p_t\xi_t(1-\tau)a_{i,t-1}+\theta_t^i \pi_t \tag{3}$$
$$(q_{t+1}+(1-\tau)p_{t+1}\xi_{t+1})a_{i,t} \geq -f_i\big[p_{t+1}(1-\delta)+r_{t+1}\big]k_{i,t+1}. \tag{4}$$

Here $f_i \in [0,1]$ is the borrowing limit of agent i. f_i is an exogenous parameter and set by law. This parameter can be viewed as an index of financial development. At date t, π_t is the firm's profit, $(\theta_t^i)_{i=1}^m$ is the exogenous share of profit with $\theta_t^i \geq 0$ for any i and t, and $\sum_{i=1}^m \theta_t^i = 1$ for any t.

In our model, consumers can borrow by using the financial asset but they face borrowing constraints. Agent i can borrow an amount if the repayment of this amount does not exceed a fraction of the market value of her (physical) capital income (including returns and depreciation). In other terms, the physical capital plays the role of collateral. The fraction f_i is less than 1 to ensure that the market value of collateral of each agent is greater than her debt. At equilibrium, the borrowing constraint (1) becomes equivalent to $q_t a_{i,t} \geq -f_i p_t k_{i,t+1}$.[5]

The government

In our model, the government levies tax on dividends and uses it to invest in research and development (R&D).

The government fixes the tax rate τ on dividends. The aggregate tax is denoted by T_t (in terms of consumption good). By construction, we have

$$T_t = \sum_{i=1}^m \tau\xi_t a_{i,t-1}.$$

Let us denote by G_t the government spending at date t. We assume that the government spending in R&D will affect the productivity of the firm at the next date. More precisely, the production function at date t is given by $F_g(G_{t-1},\cdot)$ with $F_g(G,K) = f(G)F(K)$ where f is an increasing function and $f(0)=1$. F is the original production function without government spending in R&D. When $G=0$, we recover Le Van and Pham (2015).

Firm

At date t, the representative firm takes prices (p_t, r_t) and government spending G_{t-1} as given and maximizes its profit by choosing the physical capital amount K_t.

$$(P(p_t,r_t,G_{t-1})): \quad \pi_t := \max_{K_t \geq 0} \Big[p_t F_g(G_{t-1},K_t) - r_t K_t\Big]. \tag{5}$$

[5] See Remark 2.

2.1 Equilibrium

We denote an infinite-horizon sequence of prices and quantities by

$$(p, q, r, (c_i, k_i, a_i)_{i=1}^m, K, G, T)$$

with

$$\begin{aligned}
(c_i, k_i, a_i) &:= \quad \big((c_{i,t})_{t=0}^\infty, (k_{i,t+1})_{t=0}^\infty, (a_{i,t})_{t=0}^\infty\big) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}^+,\\
(p, q, r) &:= \quad \big((p_t)_{t=0}^\infty, (q_t)_{t=0}^\infty, (r_t)_{t=0}^\infty, \big) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty,\\
(K, G, T) &:= \quad \big((K_t)_{t=0}^\infty, (G_t)_{t=0}^\infty, (T_t)_{t=0}^\infty\big) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty
\end{aligned}$$

for each $i = 1, \dots, m$.

The economy is denoted by $\mathcal{E}$ and is characterized by a list

$$\Big((u_i, \beta_i, k_{i,0}, a_{i,-1}, f_i, \theta^i)_{i=1}^m, F, f, (\xi_t)_{t=0}^\infty, \delta, \tau\Big).$$

Definition 1. *A list* $\Big(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{i=1}^m, \bar{K}_t, \bar{G}_t, \bar{T}_t\Big)_{t=0}^\infty$ *is an equilibrium of the economy* $\mathcal{E}$ *if the following conditions are met.*

(i) Price positivity: $\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing conditions:

$$\begin{aligned}
\textit{good: } & \sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) = f(\bar{G}_{t-1})F(\bar{K}_t) + (1-\tau)\xi_t,\\
\textit{capital: } & \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t},\\
\textit{financial asset: } & \sum_{i=1}^m \bar{a}_{i,t} = 1,
\end{aligned}$$

for any $t \geq 0$.

(iii) Optimal consumption plans: for any i, $(\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{t=0}^\infty$ *is a solution of the problem* $(P_i(\bar{p}, \bar{q}, \bar{r}))$.

(iv) Optimal production plan: for any $t \geq 0$, $\bar{K}_t$ *is a solution of the problem* $(P(\bar{p}_t, \bar{r}_t, \bar{G}_{t-1}))$.

(v) Government: $\bar{G}_t = \bar{T}_t$ *where* $\bar{T}_t = \sum_{i=1}^m \tau\xi_t\bar{a}_{i,t-1}$.

Remark 1. *At equilibrium, we have* $G_t = T_t = \tau\xi_t$. *Therefore, the consumption market clearing condition writes*

$$C_t + K_{t+1} + G_t = f(G_{t-1})F(K_t) + (1-\delta)K_t + \xi_t, \tag{6}$$

where $C_t := \sum_{i=1}^{m} c_{i,t}$, $K_t := \sum_{i=1}^{m} k_{i,t}$.

The output of the economy is $f(G_{t-1})F(K_t) + (1-\delta)K_t + \xi_t$ *and decomposes into three parts: private consumption* C_t*, private investment* K_{t+1} *and public investment* G_t.

By using Le Van and Pham (2015), we can prove that an equilibrium exists under the following assumptions. The detailed proof is presented in Appendix 7.

Assumption (H1). u_i is C^1, strictly increasing and concave with $u_i(0) = 0$ and $u_i'(0) = \infty$.

Assumption (H2). The function $F(\cdot)$ is C^1, strictly increasing, concave with $F(0) \geq 0$. The function $f(\cdot)$ is increasing and $f(0) = 1$.

Assumption (H3). For every $t \geq 0$ and $0 < \xi_t < \infty$.

Assumption (H4). $k_{i,0}, a_{i,-1} \geq 0$, and $(k_{i,0}, a_{i,-1}) \neq (0,0)$ for $i = 1, \ldots, m$. Moreover, $\sum_{i=1}^{m} a_{i,-1} = 1$ and $K_0 := \sum_{i=1}^{m} k_{i,0} > 0$.

Assumption (H5). The intertemporal utility functions are finite

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t) < \infty.$$

with

$$\begin{aligned} D_0 :=& \quad F_g(\xi_0, K_0) + (1-\delta)K_0 + \xi_0, \\ D_t :=& \quad F_g(\xi_{t-1}, D_{t-1}) + (1-\delta)D_{t-1} + \xi_t \quad \forall t \geq 0. \end{aligned}$$

2.2 Basis properties

Let $\big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t, G_t, T_t\big)_t$ be an equilibrium. Denote by $\mu_{i,t}$ and $\nu_{i,t+1}$ the multipliers associated to the budget and the borrowing constraint of the agent i at date t. Denote $\lambda_{i,t+1}$ the multiplier associated with constraint $k_{i,t+1} \geq 0$. We obtain

$$\beta_i^t u_i'(c_{i,t}) = p_t \mu_{i,t} \tag{7}$$

$$p_t \mu_{i,t} = (r_{t+1} + (1-\delta)p_{t+1})(\mu_{i,t+1} + f_i \nu_{i,t+1}) + \lambda_{i,t+1} \tag{8}$$

$$q_t \mu_{i,t} = (q_{t+1} + (1-\tau)p_{t+1}\xi_{t+1})(\mu_{i,t+1} + \nu_{i,t+1}). \tag{9}$$

Notice that $k_{i,t+1}\lambda_{i,t+1} = 0$ and

$$\nu_{i,t+1}\Big[\big(q_{t+1} + (1-\tau)p_{t+1}\xi_{t+1}\big)a_{i,t} + f_i\big(p_{t+1}(1-\delta) + r_{t+1}\big)k_{i,t+1}\Big] = 0.$$

The following lemma sums up the FOCs.

Lemma 1 (non-arbitrage condition)**.**

$$\frac{q_{t+1} + (1-\tau)p_{t+1}\xi_{t+1}}{q_t} = \frac{1}{\max\limits_i \big\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\big\}} \geq \frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t} \tag{10}$$

for any t*. Moreover, the inequality holds with equality if* $K_{t+1} > 0$.

Remark 2. *According to Lemma 1, we have that*

$$f_i(p_{t+1}(1-\delta)+r_{t+1})k_{i,t+1} = f_i\frac{p_t}{q_t}(q_{t+1}+(1-\tau)\xi_{t+1}p_{t+1})k_{i,t+1}. \quad (11)$$

Therefore, borrowing constraint (1) is equivalent to $q_t a_{i,t} \geq -f_i p_t k_{i,t+1}$.

3 Recession and the role of dividend taxation

We consider the specific definition of recession introduced by Le Van and Pham (2015). In Section 3.1, a more general case will be treated.

Definition 2 (recession)**.** *The productive sector experiences a recession at date* t *if no one invests in this sector, that is the aggregate capital equals zero* ($K_t = 0$)*.*

Consumers diversify their portfolio by investing in capital and the financial asset. The real return on physical capital is $\frac{r_{t+1}}{p_{t+1}}+1-\delta$, and the physical capital's maximum return is $F'(0)+1-\delta$. The real return on the financial asset is $\frac{\frac{q_{t+1}}{p_{t+1}}+(1-\tau)\xi_{t+1}}{\frac{q_t}{p_t}}$.

The following result holds.

Proposition 1 (Le Van and Pham (2015))**.** *Consider the case where* $\tau = 0$*. Assume that* $F'(0) \leq \delta$ *and there exists* $\xi > 0$ *such that* $\xi_t \geq \xi$ *for every* $t \geq 0$*. Then, there is an infinite sequence* $(t_n)_{n=0}^{\infty}$ *such that* $K_{t_n} = 0$ *for every* $n \geq 0$*.*

Proposition 1 shows that if the productivity is low in the sense that $F'(0) < \delta$, recession will appear at infinitely many dates. Since the bound ξ does not depend on technology, the cause of economic recession is no longer the financial market, but the low productivity.

Proposition 1 suggests that we should invest in R&D to improve the productivity and avoid recessions. In what follows, we will focus on the role of R&D. For simplicity, we consider a simple case where $\xi_t = \xi > 0$ for any t and $f(x) = (1+bx)^{\alpha_1}$ with $\alpha_1 > 0$. Here, the positive parameter b represents the efficiency of the R&D process.

We denote by $\rho_i \equiv 1/\beta_i - 1$ the exogenous interest rate of agent i. We have the following result.

Proposition 2. *Assume that* $\xi_t = \xi > 0$ *for any* t *and* $f(x) = (1+bx)^{\alpha_1}$ *with* $\alpha_1 > 0$*. Then,* $K_t > 0$ *if*

$$(1+b\tau\xi)^{\alpha_1}F'(0) > \delta + \max_{i=1,\dots,m}\rho_i. \quad (12)$$

Condition (12) implies that the return from the productive sector is higher than the investment cost. In this case, someone is willing to invest in the productive sector. We observe that condition (12) is satisfied if productivity ($F'(0)$), R&D efficiency and dividends (ξ) are high.

Proposition 2 has an interesting consequence. Consider a "bad" technology F (in the sense that $F'(0) < \delta$). In this case, without taxation on dividends, there is no R&D investment and the recession will arise at infinitely many dates (Proposition

1). When the government levies tax on asset dividends to finance efficient R&D (in the sense of condition (12)), the economy never falls in recession.

However, we would like also to point out that, given a low initial productivity, recession becomes unavoidable if the R&D is inefficient and dividends are low. Formally, we have.

Proposition 3. *Assume that* $\bar{\xi} := \sup_t \xi_t < \infty$ *and* $\xi := \inf_t \xi_t > 0$ *with* $f(x) = (1+bx)^{\alpha_1}$ *and* $(1+b\tau\bar{\xi})^{\alpha_1}F'(0) \leq \delta$*. Then, there exists a sequence* $(t_n)_{n=0}^{\infty}$ *such that* $K_{t_n} = 0$ *for every* $n \geq 0$*.*

Human capital

Let us introduce the human capital in the production function: $F(K)L^{\alpha_1}$. Our model can be also interpreted as an economy with exogenous labor supply $L_0 = 1$. With a government spending in human capital, the effective labor becomes $(1+bG_t)L_0$ and the marginal productivity (with respect to capital) $F'(K)(1+bG_t)^{\alpha_1}$. In this case, all the above results still hold and we would say that recession in the productive sector may be prevented if the government uses the tax on dividends to invest in human capital.

Taxes on land dividends

If $f_i = 0$ for any i, we recover exactly the asset structure of land: an agent buys land today to receive fruits (consumption good) tomorrow as land dividends and resell land thereafter. Proposition 2 shows that a good government is able to prevent recessions when land dividends are high enough. This interpretation leads to another interesting remark. Focus on a two-sector economy: agriculture (represented by land) and industry (represented by a firm). In this case, if the productivity $F'(0)$ of the industrial sector is low, the government may collect taxes on land dividends to finance R&D activities and, therefore, to improve the industrial productivity and shelter this sector from recessions. In some cases, this strategy not only avoid recession but also create more consumption good. In Section 4, the issue of optimal tax level will be addressed.

3.1 Extension: dividend taxation and growth

Consider now a more general concept of recession than Definition 2.

Definition 3. *There is a* $\bar{k}$*-recession in the productive sector at date* t *if* $K_t \leq \bar{k}$*.*

The following result generalizes Proposition 3.[6]

Proposition 4. *Assume that* $\bar{\xi} := \sup_t \xi_t < \infty$ *and* $\xi := \inf_t \xi_t > 0$ *with* $f(x) = (1+bx)^{\alpha_1}$ *and* $(1+b\tau\bar{\xi})^{\alpha_1}F'(\bar{k}) \leq \delta$*. Then, there exists a sequence* $(t_n)_{n=0}^{\infty}$ *such that* $K_{t_n} \leq \bar{k}$ *for every* $n \geq 0$*.*

[6]The proof of Proposition 4 is similar to Proposition 3.

Proposition 3 shows that $\bar{k}$-recessions will appear at infinitely many dates if b, ξ_t and productivity are low. However, the following result shows that $\bar{k}$-recessions can be prevented when dividends are high enough.

Proposition 5. *Assume that (1) $\xi_t = \xi > 0$ for any t, (2) $f(x) = (1+bx)^{\alpha_1}$ and (3) $u_i(c) = c^{1-\sigma_i}/(1-\sigma_i)$ with $\sigma \in (0,1)$. Given $\bar{k} > 0$, there exists $\bar{\xi}$ such that $K_t > \bar{k}$ for any $\xi > \bar{\xi}$ and for any $t \geq 1$.*

We may wonder whether the dividend taxation can be growth-enhancing. The next result shows the important role of dividend taxation and efficient R&D in economic growth.

Proposition 6. *Assume that (1) $\xi_t = \xi > 0$ for any t, (2) $F(K) = AK$, (3) $f(x) = (1+bx)^{\alpha_1}$ and (4) $u_i(c) = c^{1-\sigma_i}/(1-\sigma_i)$ with $\sigma \in (0,1)$. If*

$$\Big(A(1+b\tau\xi)^{\alpha_1} - \delta\Big)\Big(A(1+b\tau\xi)^{\alpha_1} + 1 - \delta\Big)^{\frac{1}{\sigma_i}-1}\beta_i^{\frac{1}{\sigma_i}} > 1 \tag{13}$$

for any i, then, $\lim\limits_{t\to\infty} K_t = \infty$.

Proposition 6 has some interesting implications. Focus on the case where the original productivity is low ($A < \delta$).

1. If there is no dividend ($\xi_t = 0$ for any t), then, according to (6), we have $K_{t+1} \leq (A+1-\delta)K_t$ for any t, which implies that $\lim\limits_{t\to\infty} K_t = 0$: the economy collapses.

2. In the case with constant positive dividend ($\xi_t = \xi > 0$ for any t), Proposition 6 suggests that, if the government levies taxes on asset dividends and invests in efficient R&D or human capital (in the sense of condition (13)), growth will be unbounded.

Notice that we have a general equilibrium approach. Hence, there does not a representative agent who chooses the level of aggregate capital K_t to maximize her intertemporal utility. So, it is not easy to prove some nice properties (in particular, monotonicity of capital stock (K_t)) as in the optimal growth theory (see Le Van and Dana (2003) among others). In this technical point of view, Proposition 6 is also relevant.

4 Optimal dividend taxation

When the government raises the tax rate τ, the net dividend $(1-\tau)\xi_t$ drops but the production level increases. It is worthy to deepen this trade-off by considering the optimal taxation on dividends. To this purpose, we assume that the government chooses $\tau \in [\underline{\tau}, \bar{\tau}] \subset [0,1]$, where $\underline{\tau}$ and $\bar{\tau}$ are exogenous parameters,[7] in order to maximize the aggregate consumption at the steady state. Let us start by defining the steady state formally.

[7] The exogenous parameters $\underline{\tau}$ and $\bar{\tau}$ represent political or institutional constraints that we do not microfound here.

Definition 4. *Assume that* $\xi_t = \xi > 0$ *and* $\tau_t = \tau \in [0,1]$ *for any* t. *A steady state is an equilibrium* $\big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i=1}^m, K_t, G_t, T_t\big)_t$ *such that* $p_t = 1$, $q_t = q, r_t = r$, $c_{i,t} = c_i, k_{i,t} = k_i$ *and* $a_{i,t} = a_i$ *for any* i *and* t, *and* $K_t = K$, $G_t = G$ *and* $T_t = T$ *for any* t.

In this definition, the sequence of consumption prices is normalized ($p_t = 1$ for any t). We provide now sufficient conditions for steady state uniqueness.

Lemma 2. *Let* $\beta_1 > \beta_i$ *for any* $i \geq 2$ *and* $f_i < 1$ *for any* i. *Assume also that* $\xi_t = \xi, \tau_t = \tau \in [0,1]$ *for any* t *and that* F *is strictly concave and linear with* $F'(0) = \infty$. *Then, there is a unique steady state:*

$$1 = \beta_1\Big(f(\tau\xi)F'(K) + 1 - \delta\Big) \tag{14}$$

$$r = f(\tau\xi)F'(K) \text{ and } q = \frac{(1-\tau)\xi\beta_1}{1-\beta_1} \tag{15}$$

$$k_1 = K,\ a_i = 1 \text{ and } c_1 = (r-\delta)K + \theta_1\pi + (1-\tau)\xi \tag{16}$$

$$a_i = k_i = 0 \text{ and } c_i = \theta_i\pi \text{ for } i = 2,\dots,m. \tag{17}$$

Since $\beta_1 > \beta_i$ for any $i = 2,\dots,m$, the borrowing constraints of any consumer $i = 2,\dots,m$ are binding. Moreover, the condition $f_i < 1$ for any i implies that no agent $i = 2,\dots,m$ will invest in physical capital.[8] Hence, the income of any agent $i = 2,\dots,m$ equals their profit share.[9]

Since the aggregate capital level K is determined by (14) and F is strictly concave, we see that K is uniquely determined. Moreover, we also see that K is increasing in β_1, τ and ξ, and decreasing in δ.

For simplicity, in what follows, we write β instead of β_1.

The aggregate consumption is given by

$$C = (1-\tau)\xi + f(\tau\xi)F(K) - \delta K.$$

For the sake of simplicity, we consider a Cobb-Douglas production function $F(K) = AK^\alpha$ with $\alpha \in (0,1)$. In this case, we have

$$K = \Big(\frac{\alpha A f(\tau\xi)}{\frac{1}{\beta} + \delta - 1}\Big)^{\frac{1}{1-\alpha}}. \tag{18}$$

The aggregate consumption is given by

$$C = f(\tau\xi)AK^\alpha - \delta K + (1-\tau)\xi = B_1\Big(Af(\tau\xi)\Big)^{\frac{1}{1-\alpha}} + (1-\tau)\xi \tag{19}$$

[8]If $f_i = 1$ for any i, there may be an equilibrium indeterminacy (in term of assets held by agents).

[9]Notice that, when there are at least 2 agents, say 1 and 2, whose rates of time preference are $\beta_1 = \beta_2 > \beta_i$ for any $i = 3,\dots,m$, the aggregate capital stock K remains unique and still determined by (14) but their income distribution depends on their initial distribution of capital.

where $B_1 := \alpha^{\frac{\alpha}{1-\alpha}} \dfrac{\frac{1}{\beta} - 1 + \delta(1-\alpha)}{\left(\frac{1}{\beta} - 1 + \delta\right)^{\frac{1}{1-\alpha}}}$.

The government's problem writes

$$\max_{\tau \in [\underline{\tau}, \bar{\tau}]} B_1 \big(Af(\tau\xi)\big)^{\frac{1}{1-\alpha}} + (1-\tau)\xi$$

or, if $f(\tau\xi) = (1 + b\xi\tau)^{\alpha_1}$, more explicitly,

$$\max_{\tau \in [\underline{\tau}, \bar{\tau}]} \left[B_1 A^{\frac{1}{1-\alpha}} (1 + b\xi\tau)^{\sigma} - \xi\tau\right] \tag{20}$$

where now $\sigma := \frac{\alpha_1}{1-\alpha}$. If $\alpha_1 < 1 - \alpha$, then, $\sigma < 1$, which implies in turn that the objective function in (20) is strictly concave.[10] By consequence, we obtain the following result.

Proposition 7. *Let $F(K) = AK^{\alpha}$ and $f(x) = (1 + bx)^{\alpha_1}$ with $\alpha + \alpha_1 < 1$. There are three possibilities.*

1. *If $\sigma b B_1 A^{\frac{1}{1-\alpha}} \geq (1 + b\bar{\tau}\xi)^{1-\sigma}$, then $\tau^* = \bar{\tau}$.*
2. *If $\sigma b B_1 A^{\frac{1}{1-\alpha}} \leq (1 + b\underline{\tau}\xi)^{1-\sigma}$, then $\tau^* = \underline{\tau}$.*
3. *If $(1 + b\underline{\tau}\xi)^{1-\sigma} < \sigma b B_1 A^{\frac{1}{1-\alpha}} < (1 + b\bar{\tau}\xi)^{1-\sigma}$, then τ^* is the solution of the following equation*
$$\sigma b B_1 A^{\frac{1}{1-\alpha}} = (1 + b\tau\xi)^{1-\sigma}.$$

Comparative statics

Consider the role of parameters b and A that represent R&D efficiency and the original TFP. Proposition 7 shows that when R&D efficiency b and TFP A are very high (in the sense of the first point in Proposition 7), the optimal tax rate equals $\bar{\tau}$, the highest affordable tax rate. But, when b and A are low (enough), the optimal tax rate equals $\underline{\tau}$ and the government implements the lowest taxation.

The following result is immediate.

Corollary 1. *In the third case of Proposition 7, the optimal level τ^* is increasing in β, A and b, but decreasing in ξ.*

Remark 3. *When the government objective is a measure of welfare such as the aggregation of agents' intertemporal utilities, it is difficult to find closed solutions. Indeed, because of the financial market imperfections, it may become impossible to provide a closed form for equilibrium prices: given a tax rate τ, the equilibrium may fail to be unique (see Proposition 10). Even in the case of uniqueness, equilibrium allocations and prices may fail to be smooth in τ and the government's maximization problem becomes a hopeless challenge.*[11]

[10] If $\alpha_1 \geq 1 - \alpha$, the objective function turns out to be convex and the solution becomes either $\underline{\tau}$ or $\bar{\tau}$.

[11] Differently from Chamley (1986) and Judd (1985).

5 Dividend taxation and bubbles in asset prices

Consider the case where the tax rate on dividends is no longer constant over time and focus on the impact of a tax sequence (τ_t) on bubbles in asset prices. Before starting, a definition of asset bubble is needed.

Lemma 1 still holds with non-stationary tax rates τ_t and the following asset pricing as well:

$$\frac{q_t}{p_t} = \gamma_{t+1}(\frac{q_{t+1}}{p_{t+1}} + (1-\tau_{t+1})\xi_{t+1})$$

where $\gamma_{t+1} := \max_i \frac{\beta_t u_i(c_{i,t+1})}{u_i(c_{i,t})}$ is the discount factor of the economy from date t to date $t+1$.

Then, we can decompose the asset price q_0/p_0 (in term of consumption good at the initial date) into two parts:

$$\frac{q_0}{p_0} = \sum_{t=1}^{\infty} Q_t(1-\tau_t)\xi_t + \lim_{T\to\infty} Q_T \frac{q_T}{p_T}$$

where $Q_t := \prod_{s=1}^{t} \gamma_t$ is the discount factor of the economy from the initial date to date t. Following Kocherlakota (1992), Santos and Woodford (1997), we introduce the notion of bubble.

Definition 5. $\sum_{t=1}^{\infty} Q_t(1-\tau_t)\xi_t$ *is the financial asset fundamental value. Bubbles exist at equilibrium if the financial asset real price exceeds its fundamental value:* $\frac{q_0}{p_0} > \sum_{t=1}^{\infty} Q_t(1-\tau_t)\xi_t$.

Apply the same argument by Montrucchio (2004) and Le Van et al. (2014) to characterize the existence of bubbles.

Proposition 8. *The following statements are equivalent.*

1. *Bubbles exist.*
2. $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} > 0$.
3. $\sum_{t\geq 1} \frac{(1-\tau_t)\xi_t}{q_t} < \infty$.

The following proposition provides conditions (using exogenous parameters) under which bubbles are ruled out.

Proposition 9. *Let* $\xi_t = \xi > 0$ *for any* t. *If* $f(\xi)F'(\infty) < \delta$ *and* $\limsup_t \tau_t < 1$ *for any* t*, then bubbles are ruled out.*

Proposition 9 suggests that a bubble in financial asset is entailed by a tax rate τ_t sufficiently close to 1. We may wonder whether bubbles exist when $\lim_{t\to\infty} \tau_t = 1$. The answer is articulated through some examples.

5.1 Examples of bubbles: the role of dividend taxation

Assume that there are 2 consumers H and F. Let $u_i(c) = ln(c)$, $\beta_i = \beta \in (0,1)$ and $f_i = 0$ for $i = \{H, F\}$ with $\delta \in (0,1)$. Agents' initial endowments are given by $k_{H,0} = 0$, $a_{H,-1} = 0$, $k_{F,0} > 0$ and $a_{F,-1} = 1$, while their profit shares by:

$$\begin{aligned}\left(\theta_{2t}^H, \theta_{2t+1}^H\right) &= (1,0) \\ \left(\theta_{2t}^F, \theta_{2t+1}^F\right) &= (0,1)\end{aligned}$$

for any $t \geq 0$.

Focus on a linear production function: $F(K) = AK + B$, where $A, B > 0$ and $\beta(1 - \delta + f(\xi)A) \leq 1$.[12] This production function can be viewed as a particular case of the function $F(K, L) = AK + BL$ with inelastic labor supply (equal to one).

Notice that $F_g(G_{t-1}, K_t) = f(\tau_{t-1}\xi)(AK_t + B)$ and $\pi_t = f(\tau_{t-1}\xi)B$ for any t.

Let us now construct an equilibrium.

The allocations of consumer H are given by

$$k_{H,2t} = 0, a_{H,2t-1} = 0 \tag{21}$$
$$c_{H,2t-1} = (1 - \delta + r_{2t-1})K_{2t-1} + q_{2t-1} + (1 - \tau_{2t-1})\xi_{2t-1} \tag{22}$$
$$k_{H,2t+1} = K_{2t+1}, a_{H,2t} = 1 \tag{23}$$
$$c_{H,2t} = \pi_{2t} - K_{2t+1} - q_{2t} \tag{24}$$

while the allocations of consumer F by

$$k_{F,2t} = K_{2t}, a_{F,2t} = 1 \tag{25}$$
$$c_{F,2t-1} = \pi_{2t-1} - K_{2t} - q_{2t-1} \tag{26}$$
$$k_{F,2t+1} = 0, a_{F,2t} = 0 \tag{27}$$
$$c_{F,2t} = (1 - \delta + r_{2t})K_{2t} + q_{2t} + (1 - \tau_{2t})\xi_{2t}. \tag{28}$$

Prices and the aggregate capital solve the following system: for any t,

$$K_{t+1} + q_t = \frac{\beta}{1+\beta}(F_t(K_t) - r_t K_t) = \frac{\beta f(\tau_{t-1}\xi)B}{1+\beta} \tag{29}$$
$$q_{t+1} + (1 - \tau_{t+1})\xi_{t+1} = q_t(r_{t+1} + 1 - \delta) \tag{30}$$
$$q_t, K_t > 0 \tag{31}$$

with

$$p_t = 1 \text{ and } r_t = f(\tau_{t-1}\xi)A$$

This sequence of allocations and prices is an equilibrium since it satisfies all the conditions of Lemma 3 in Appendix 7.1.

When $\tau_t < 1$ for any t, the asset price q_t is strictly positive. In this case, there exists an equilibrium bubble if and only if $\sum_{t=1}^{\infty} \frac{(1-\tau_t)\xi_t}{q_t} < \infty$. Since $\xi_t = \xi$ for any

[12] Condition $\beta(1 - \delta + f(\xi)A) \leq 1$ ensures that FOCs are satisfied. This and condition (12) are not mutually exclusive since (12) implies $K_t > 0$. However, in some cases, we do not need (12) to have $K_t > 0$.

t, a bubble exists if and only if $\sum_{t=1}^{\infty} \frac{(1-\tau_t)}{q_t} < \infty$. Condition $q_t \leq \frac{\beta f(\xi)B}{1+\beta}$ implies that a bubble exists only if $\sum_{t=1}^{\infty}(1-\tau_t) < \infty$. By consequence, bubbles are ruled out if $\limsup_{t\to\infty} \tau_t < 1$. We recover the result in Proposition 9.

Consider now the case where τ_t can tend to 1. The asset fundamental value is equal to

$$FV := \sum_{s=1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+f(\xi\tau_0)A)\cdots(1-\delta+f(\xi\tau_{s-1})A)}$$

It is easy to see that FV is decreasing in each τ_t and $FV \leq \sum_{s=1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+f(\xi)A)^t}$. Therefore, we can choose (τ_t) such that $FV < B_0 := \frac{\beta f(\tau_{-1}\xi)B}{1+\beta}$. For example, we can choose $\tau_t = 1 - x^t$ with $x < \min(1-\delta+f(\xi)A, 1)$.

Choose $q_0 > FV$ and (q_t) such that

$$q_0 = \sum_{s=1}^{t} \frac{\xi(1-\tau_s)}{(1-\delta+f(\xi\tau_0)A)\cdots(1-\delta+f(\xi\tau_{s-1})A)} + \frac{q_t}{(1-\delta+f(\xi\tau_0)A)\cdots(1-\delta+f(\xi\tau_{t-1})A)}. \tag{32}$$

The above sequence is an equilibrium price if $q_t < \frac{\beta f(\tau_{t-1}\xi)B}{1+\beta}$ for any t (this condition implies $K_{t+1} > 0$). This inequality holds if $q_0 < B_0$ and $1-\delta+f(\xi)A \leq 1$. In this case, bubbles exist.

Summing up, we get the following result.

Proposition 10 (continuum of bubbly equilibria)**.** *Let $f(\xi)A \leq \delta$ and $(\tau_t)_t$ such that*

$$FV := \sum_{s=1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+f(\xi\tau_0)A)\cdots(1-\delta+f(\xi\tau_{s-1})A)} < B_0.$$

Then, any (q_t) with $q_0 \in (FV, B_0)$ and $(q_t)_{t\geq 1}$ satisfying (32) is a bubbly price sequence.

The price sequence (q_t) with $q_0 = FV$ and $(q_t)_{t\geq 1}$ satisfying (32) is bubbleless.

Let us consider some consequences of Proposition 10.

Corollary 2. *Let the tax sequence satisfy $1-\tau_t = x^t$ where $x > 0$. If x is small enough, there exists a bubbly equilibrium.*

The intuition is that, when the tax rates τ_t tend to 1, the after-tax dividend tends to zero and the financial asset fundamental value may turn out to be lower than its price. In this case, an asset bubble arises.

Since $q_0 \geq FV$, Proposition 10 also shows that FV is the minimum level above which q_0 is an equilibrium price with bubbles. It is easy to see that FV is decreasing in each ξ_t. Thus, we concludes that bubbles are more likely to appear when sequence of tax receipts ξ_t increases.

Corollary 3. *If $A \leq \delta$ and $\sum_{s=1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+A)^s} < B_0$, then any price sequence $(q_t)_{t\geq 0}$ determined by (32) with $q_0 \in (\sum_{s=1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+A)^s}, B_0)$, is a sequence of prices with bubbles.*

We observe that, for any form of function f, there is a bubbly equilibrium. It means that when the taxes on dividends are sufficiently high, bubbles may exist whatever the level of R&D efficiency. In our example, the key factor for the existence of bubbles is the level of after-tax dividend $\xi(1-\tau_t)$.

Remark 4 (asset price and dividend taxation)**.** *In Proposition 10, let $q_0 = FV + \bar{d}$ with $\bar{d} \in (0, B_0 - FV)$, and then bubbles arise. According to (32), we can compute the asset price at date t as follows*

$$\begin{aligned} q_t = &(1-\delta+f(\xi\tau_0)A)\cdots(1-\delta+f(\xi\tau_{t-1})A)\bar{d} \\ &+ \sum_{s=t+1}^{\infty} \frac{\xi(1-\tau_s)}{(1-\delta+f(\xi\tau_t)A)\cdots(1-\delta+f(\xi\tau_{s-1})A)} \end{aligned} \tag{33}$$

It is easy to see that q_t is increasing τ_s for any $s \leq t-1$ but decreasing in τ_s for any $s \geq t$.

6 Conclusion

We have proved that a low productivity entails recessions at infinitely many dates. However, when the government taxes the consumers' dividends and spend this fiscal revenue to invest in R&D activities, the productivity of firms is enhanced and recession may be avoided. Thus happens if: (1) the R&D process is efficient or (2) dividends are high. The economy may grow without bounds when the R&D process becomes very efficient.

The optimal level of dividend taxation increases in the R&D efficiency the TFP, but decreases in the level of dividends.

Given stationary technology and dividend, asset bubbles are ruled out if the taxes on asset dividends are not sufficiently high (in the sense that $\limsup_t \tau_t < 1$). However, when τ_t tends to 1, there may be a room for asset bubbles. Interestingly, asset bubbles may appear whatever the efficiency of the R&D process.

7 Appendix: formal proofs

The existence of equilibrium. We consider the intermediate economy $\tilde{\mathcal{E}}$ as the economy $\mathcal{E}$ but the government is not taken into account. Denote $\tilde{\xi}_t := (1-\tau)\xi_t$ and the function $\tilde{F}_t$ defined by $\tilde{F}_t(K) := F_g(\tau\xi_{t-1}, K)$. According to Le Van and Pham (2015), there exists an equilibrium $\left(\tilde{p}_t, \tilde{q}_t, \tilde{r}_t, (\tilde{c}_{i,t}, \tilde{k}_{i,t+1}, \tilde{a}_{i,t})_{i=1}^m, \tilde{K}_t\right)_{t=0}^{\infty}$ of the economy $\tilde{\mathcal{E}}$, i.e., the following conditions hold:

1. $\tilde{p}_t, \tilde{q}_t, \tilde{r}_t > 0$ for $t \geq 0$.

2. For any $t \geq 0$,

$$\sum_{i=1}^{m} (\tilde{c}_{i,t} + \tilde{k}_{i,t+1} - (1-\delta)\tilde{k}_{i,t}) = \tilde{F}_t(\tilde{K}_t) + (1-\tau)\xi_t \tag{34}$$

$$\tilde{K}_t = \sum_{i=1}^{m} \tilde{k}_{i,t} \tag{35}$$

$$\sum_{i=1}^{m} \bar{a}_{i,t} = 1. \tag{36}$$

3. Optimal consumption plans: for any i, $(\tilde{c}_{i,t}, \tilde{k}_{i,t+1}, \tilde{a}_{i,t})_{t=0}^{\infty}$ is a solution of the problem $(P_i(\tilde{p}, \tilde{q}, \tilde{r}))$.

4. Optimal production plan: for any $t \geq 0$, $\bar{K}_t$ is a solution of the following problem

$$\max_{K_t \geq 0} \Big[\tilde{p}_t \tilde{F}_t(\tilde{K}_t) - \tilde{r}_t \tilde{K}_t\Big]. \tag{37}$$

It is easy to see that $\Big(\tilde{p}_t, \tilde{q}_t, \tilde{r}_t, (\tilde{c}_{i,t}, \tilde{k}_{i,t+1}, \tilde{a}_{i,t})_{i=1}^{m}, \tilde{K}_t, \tau\xi_t, \tau\xi_t\Big)_{t=0}^{\infty}$ is an equilibrium the economy $\mathcal{E}$.

□

Proof for Proposition 2. If $K_{t+1} = 0$, we have

$$\begin{aligned}
\sum_{i=1}^{m} c_{i,t} &= F(K_t) + (1-\delta)K_t + (1-\tau)\xi, \\
\sum_{i=1}^{m} c_{i,t+1} + K_{t+2} &= F(0) + (1-\tau)\xi.
\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{m} c_{i,t} \geq F(0) + (1-\tau)\xi \geq \sum_{i=1}^{m} c_{i,t+1}. \tag{38}$$

Consequently, there exists $i \in \{1, \cdots, m\}$ such that $c_{i,t} \geq c_{i,t+1}$, hence $u_i'(c_{i,t+1}) \geq u_i'(c_{i,t})$. Thus, we have that

$$\frac{1}{(1+b\tau\xi)^{\alpha_1}F'(0) + 1 - \delta} \geq \max_j \frac{\beta_j u_j'(c_{j,t+1})}{u_j'(c_{j,t})} \quad \geq \quad \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \geq \beta_i$$

So $1 \geq \big((1+b\tau\xi)^{\alpha_1}F'(0) + 1 - \delta\big)\beta_i$, contradiction! □

Proof for Proposition 3. We claim that there exists an infinite increasing sequence $(t_n)_{n=0}^{\infty}$ such that $\frac{q_{t_n}}{p_{t_n}} + (1-\tau)\xi_{t_n} > \frac{q_{t_n-1}}{p_{t_n-1}}$ for every $n \geq 0$.
Indeed, if not, there exists t_0 such that $\frac{q_{t+1}}{p_{t+1}} + (1-\tau)\xi_{t+1} \leq \frac{q_t}{p_t}$ for every $t \geq t_0$.

Combining with $\xi_t \geq \xi$ for every $t \geq 0$ and by using induction argument, we can easily prove that

$$\frac{q_{t_0}}{p_{t_0}} \geq \frac{q_{t+t_0}}{p_{t+t_0}} + t(1-\tau)\xi$$

for every $t \geq 0$. Let $t \to \infty$, we have $\frac{q_{t_0}}{p_{t_0}} = \infty$, contradiction![13]

Therefore, there exists a sequence (t_n) such that for every $n \geq 0$, $\frac{q_{t_n}}{p_{t_n}} + (1-\tau)\xi_{t_n} > \frac{q_{t_n-1}}{p_{t_n-1}}$. Therefore, by assumptions in Proposition 3, we have

$$\frac{\frac{q_{t_n}}{p_{t_n}} + (1-\tau)\xi_{t_n}}{\frac{q_{t_n-1}}{p_{t_n-1}}} > 1 \geq (1 + b\tau\bar{\xi})^{\alpha_1} F'(0) + 1 - \delta.$$

Assume that $K_{t_n} > 0$. According to Lemma 1, we see that

$$\frac{\frac{q_{t_n}}{p_{t_n}} + (1-\tau)\xi_{t_n}}{\frac{q_{t_n-1}}{p_{t_n-1}}} = (1 + b\tau\xi_{t_n})^{\alpha_1} F'(K_{t_n}) + 1 - \delta < (1 + b\tau\bar{\xi})^{\alpha_1} F'(0) + 1 - \delta.$$

This is a contradiction. Therefore, $K_{t_n} = 0$ for any n.

$\square$

Proof for Proposition 5. We see that

$$\frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})} \leq \max_j \frac{\beta_j u_j'(c_{j,t})}{u_j'(c_{j,t-1})} \leq \frac{1}{(1 + b\tau\xi)^{\alpha_1} F'(K_t) + 1 - \delta}.$$

Let us denote $B_i := \big(f(\tau\xi)F'(K_t) + 1 - \delta\big)\beta_i$. The above inequality implies that $c_{i,t} \geq x_i c_{i,t-1}$ for any i, where $x_i = B_i^{\frac{1}{\sigma_i}}$ depends increasingly in $b\xi$ and $F'(K_t)$. Moreover, we have x_i tends ton infinity when b tends to infinity.

Denote

$$x := \min_i x_i = \min_i \Big[\big(f(\tau\xi)F'(K_t) + 1 - \delta\big)\beta_i\Big]^{\frac{1}{\sigma_i}}.$$

Since x depends on $b\xi$ and K_t, we write $x = x(b\xi, K_t)$. x increases in $b\xi$ but decreases in K_t.

We have $C_t \geq xC_{t-1}$. By market clearing conditions, we have

$$C_{t-1} + K_t = f(G_{t-2})F(K_{t-1}) + (1-\delta)K_{t-1} + (1-\tau)\xi \quad (39)$$
$$C_t + K_{t+1} = f(G_{t-1})F(K_t) + (1-\delta)K_t + (1-\tau)\xi. \quad (40)$$

and hence

$$\begin{aligned} f(\tau\xi)F(K_t) + (1-\delta)K_t + (1-\tau)\xi &\geq f(\tau\xi)F(K_t) + (1-\delta)K_t + (1-\tau)\xi - K_{t+1} \\ &\geq x\Big(f(\tau\xi)F(K_{t-1}) + (1-\delta)K_{t-1} + (1-\tau)\xi - K_t\Big). \end{aligned}$$

[13] Our result is still valid if the condition "$\xi_t \geq \xi > 0$ for every $t \geq 0$" is replaced by "$\sum_{t=0}^{\infty} \xi_t = \infty$".

This implies that

$$\frac{f(\tau\xi)F(K_t)+(1-\delta)K_t+(1-\tau)\xi}{x}+K_t \geq f(\tau\xi)F(K_{t-1})+(1-\delta)K_{t-1}+(1-\tau)\xi$$
$$\geq (1-\tau)\xi.$$

Thus, we get

$$\frac{f(\tau\xi)F(K_t)}{x}+\frac{(1-\delta)K_t}{x}+K_t \geq (1-\tau)\xi\frac{x-1}{x}. \tag{41}$$

Since x decreases in K_t, we have: for each $\xi>0$, there exists a unique $K(\xi)$ such that

$$\frac{f(\tau\xi)F(K(\xi))}{x}+\frac{(1-\delta)K(\xi)}{x}+K(\xi) = (1-\tau)\xi\frac{x-1}{x}. \tag{42}$$

Since $\sigma_i<1$ for any i, we see that $\dfrac{f(\tau\xi)}{x}$ is decreasing in ξ. Combining with the fact that x is increasing in ξ, we get that $K(\xi)$ is increasing in ξ. Moreover, we see that $\lim\limits_{\xi\to\infty} K(\xi)=\infty$. As a result, there exists $\bar{\xi}>0$ such that $K(\bar{\xi})>\bar{k}$ for any t. Therefore, $K_t>\bar{k}$ for any t. □

Proof for Proposition 6. We see that

$$\frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})} \leq \max_j \frac{\beta_j u_j'(c_{j,t})}{u_j'(c_{j,t-1})} \leq \frac{1}{(1+b\tau\xi)^{\alpha_1}F'(K_t)+1-\delta} = \frac{1}{(1+b\tau\xi)^{\alpha_1}A+1-\delta}.$$

Let us denote $B_i := \big(f(\tau\xi)A+1-\delta\big)\beta_i$.

The above inequality implies that $c_{i,t}\geq x_i c_{i,t-1}$ for any i, where $x_i=B_i^{\frac{1}{\sigma_i}}$ depends increasingly in $b\xi$ and A. Moreover, we have x_i tends to infinity when b tends to infinity.

Denote

$$x := \min_i x_i = \min_i \Big[\big(f(\tau\xi)A+1-\delta\big)\beta_i\Big]^{\frac{1}{\sigma_i}}.$$

Using the same argument in the proof of Proposition 5, we have

$$\frac{f(\tau\xi)F(K_t)+(1-\delta)K_t+(1-\tau)\xi}{x}+K_t \geq f(\tau\xi)F(K_{t-1})+(1-\delta)K_{t-1}+(1-\tau)\xi.$$

This implies that

$$K_t\Big(f(\tau\xi)A+1-\delta+x\Big) \geq K_{t-1}\Big(f(\tau\xi)A+1-\delta\Big)x+(x-1)(1-\tau)\xi. \tag{43}$$

Thus, we get that

$$\frac{K_t}{K_{t-1}} \geq \frac{\Big(f(\tau\xi)A+1-\delta\Big)x}{f(\tau\xi)A+1-\delta+x}. \tag{44}$$

We see that $\frac{\Big(f(\tau\xi)A+1-\delta\Big)x}{f(\tau\xi)A+1-\delta+x} > 1$ if and only if

$$\Big(f(\tau\xi)A-\delta\Big)x > f(\tau\xi)A+1-\delta \tag{45}$$

$$\Longleftrightarrow \Big(f(\tau\xi)A-\delta\Big)\Big(f(\tau\xi)A+1-\delta\Big)^{\frac{1}{\sigma_i}-1}\beta_i^{\frac{1}{\sigma_i}} > 1 \quad \forall i. \tag{46}$$

This is satisfied thank to condition (13). By consequence, we obtain that $\lim_{t\to\infty} K_t = \infty$.

Remark 5. *The condition (46) do not violate Assumption (H5). Indeed, we have, for any t,*

$$D_{t+1} = \big(f(\tau\xi)A+1-\delta\big)D_t + \xi \tag{47}$$

Given $f(\tau\xi)A-\delta > 0$, we see that $\lim_{t\to\infty} \frac{D_t}{\big(f(\tau\xi)A+1-\delta\big)^t} < \infty$. So, Assumption (H5) is satisfied if

$$\sum_t \beta_i^t\big(f(\tau\xi)A+1-\delta\big)^{t(1-\sigma_i)} < \infty \tag{48}$$

This condition is satisfied if $\beta_i\Big(f(\tau\xi)A+1-\delta\Big)^{1-\sigma_i} < 1$ which does not contradict condition (46).

□

Proof of Lemma 2. Let $\big(p, q, r, (c_i, k_i, a_i)_{i=1}^m, K, G, T\big)$ be a steady state equilibrium. By FOCs, there exists $x_i \geq 0$, and $y_i \geq 0$ such that

$$1 = (r+1-\delta)(\beta_i + f_i x_i) + y_i \tag{49}$$

$$q = (q + (1-\tau)\xi)(\beta_i + x_i) \tag{50}$$

$$k_i y_i = 0, \quad x_i\big((q+(1-\tau)\xi)a_i + f_i(1-\delta+r)k_i\big) = 0. \tag{51}$$

According to (50) and $\beta_1 > \beta_i$ for any $i \geq 2$, we have $x_1 = 0$ and $x_i > 0$ for any $i \geq 2$ which implies that $(q+(1-\tau)\xi)a_i + f_i(1-\delta+r)k_i = 0$ for any $i \geq 2$.

Since $F'(0) = \infty$, we have $r+1-\delta = \frac{q+(1-\tau\xi)}{q} = \frac{1}{\beta_i+x_i}$. According to (49), we obtain that, for any i,

$$1 = \frac{\beta_i + f_i x_i}{\beta_i + x_i} + y_i \tag{52}$$

For each $i \geq 2$, since $x_i > 0$, and $f_i < 1$, we obtain that $y_i > 0$. Therefore, we get that $k_i = 0$, and hence $a_i = 0$ for each $i \geq 2$. So, we can compute $c_i = \theta_i\pi$ for each $i \geq 2$.

Since $F'(0) = \infty$ we have $K > 0$, so $k_1 = K > 0$. According to (49), we see that K is determined by

$$1 = \Big(f(\xi\tau)F'(K) + 1 - \delta\Big)\beta_1. \tag{53}$$

It is now easy to obtain that $a_i = 1$ and $c_1 = (r-\delta)K + \theta_1\pi + (1-\tau)\xi$. □

Proof of Proposition 9. According to (6), we have

$$C_t + K_{t+1} + G_t = f(G_{t-1})F(K_t) + (1-\delta)K_t + \xi_t \tag{54}$$
$$= f((1-\tau_{t-1})\xi_{t-1})F(K_t) + (1-\delta)K_t + \xi_t \tag{55}$$
$$\leq f(\xi)F(K_t) + (1-\delta)K_t + \xi. \tag{56}$$

Therefore, $K_{t+1} < f(\xi)F(K_t) + (1-\delta)K_t + \xi$. Since $f(\xi)F'(\infty) < \delta$, it is easy to prove that the capital stock (K_t) is uniformly bounded from above.

By using $\sum_{t=1}^{\infty} Q_t(1-\tau_t)\xi < \infty$ and $\limsup_t \tau_t < 1$, we get that $\sum_{t=1}^{\infty} Q_t < \infty$ and hence $\lim_{t\to\infty} Q_t = 0$.

Since (K_t) is uniformly bounded from above we have $\lim_{T\to\infty} Q_T k_{i,T+1} = 0$ for any i, and

$$\sum_{t=1}^{\infty} f(\tau_t\xi)F(K_t)Q_t \leq \sum_{t=1}^{\infty} f(\xi)F(K_t)Q_t < \infty.$$

We can prove that there is no financial asset bubble by using the argument in the proof of Proposition 8 in Le Van and Pham (2015). □

7.1 A sufficient condition for the equilibrium

Let us denote $I := \{1, 2, \ldots, m\}$. We give sufficient conditions for a sequence $\big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i\in I}, K_t, G_t, T_t\big)_t$ to be an equilibrium. The utility may satisfy $u_i(0) = -\infty$.

Lemma 3. *Let $f_i = 0$ for any i. A sequence $\big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t})_{i\in I}, K_t, G_t, T_t\big)_t$ is an equilibrium, if the sequence $\big(p_t, q_t, r_t, (c_{i,t}, k_{i,t+1}, a_{i,t}, \zeta_{i,t}, \epsilon_{i,t})_{i\in I}, K_t, G_t, T_t\big)_t$ satisfies the following conditions.*

(i) For any i and t, $c_{i,t} > 0$, $k_{i,t+1} \geqslant 0$, $a_{i,t} \geqslant 0$, $\zeta_{i,t} \geqslant 0$ and $\epsilon_{i,t} \geqslant 0$.

For any t, $p_t = 1$, $q_t > 0$ and $r_t > 0$.

(ii) The first-order conditions hold

$$\frac{1}{r_{t+1} + 1 - \delta} = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \zeta_{i,t}$$
$$\frac{q_t}{q_{t+1} + (1-\tau_{t+1})\xi_{t+1}} = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \epsilon_{i,t}$$

with $\zeta_{i,t} k_{i,t+1} = 0$ and $\epsilon_{i,t} a_{i,t} = 0$.

(iii) The transversality conditions are satisfied:

$$\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) k_{i,t+1} = \lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t a_{i,t} = 0.$$

(iv) For any t, $F_g(G_{t-1}, K_t) - r_t K_t = \max_{K: K\geq 0} \{F_g(G_{t-1}, K) - r_t K\}$.

(v) For any t, $c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} + q_t a_{i,t} = r_t k_{i,t} + (q_t + (1-\tau_t)\xi_t)a_{i,t-1} + \theta^i_t \pi_t$ where $\pi_t = F_g(G_{t-1}, K_t) - r_t K_t$.

(vi) For any t, $K_t = \sum_{i\in I} k_{i,t}$.

(vii) For any t, $\sum_{i\in I} a_{i,t} = 1$.

(viii) For any t, $G_t = T_t = (1-\tau_t)\xi_t$.

Proof. The proof is left to the reader. □

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