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Hubert JAYET
Hubert.jayet@univ-lille1.fr - LEM UMR9221

Léa MARCHAL
lea.marchalbatut@gmail.com - LEM UMR9221

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Complementarity and substitution between factor flows in a North-South framework

Hubert JAYET[*], Léa MARCHAL[†]

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Abstract

This paper provides new theoretical evidence about the linkages existing between migration and foreign direct investment flows. This new theoretical contribution aims at filling a gap between trade theory and available empirical evidence on the migration-FDI nexus. We develop a general equilibrium model resting upon the standard Heckscher-Ohlin (1919, 1933) framework. We consider three internationally mobile factors (capital, unskilled and skilled labor) and two transportable goods, both internationally traded. We assume a developing economy, amply endowed with unskilled labor and poorly endowed with skilled labor; and a developed economy, poorly endowed with unskilled labor and well endowed with skilled labor. We start examining what happens when the whole capital stock is invested in the North. Then, we look at the changes induced by an exogenous transfer of capital from the North to the South. Finally, we introduce imperfect international mobility of factors: investors ask for a risk premium for moving capital to the North and migrant workers must cover a migration cost. Looking at flows generated by changes in the risk premium and migration costs, we find a relation of substitution between capital and unskilled labor flows, and a relation of complementarity between capital and skilled labor flows.

Key words: International migration, Skilled and Unskilled workers, FDI, Trade
JEL Classifications: F11, F21, F22, J61

[*]LEM-CNRS UMR 9221, University of Lille, hubert.jayet@univ-lille1.fr
[†]LEM-CNRS UMR 9221, University of Lille, lea.marchal@ed.univ-lille1.fr

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1 Introduction

This paper provides new theoretical evidence about the linkages existing between migration and foreign direct investment (FDI) flows. We aim at contributing to fill a gap between trade theory and available empirical evidence about the relations between migration and FDI. It is usually considered that traditional trade theory leads to the conclusion that migration and FDI flows must be substitute: firms looking for workers available outside their home country can attract them, generating inward migration, or invest in the countries where workers are available, generating outward FDI flows. Then, migration and FDI tend to move in opposite directions.

This theoretical prediction is not confirmed by empirical evidence. Empirical analysis carried out so far has led to three main results. First, skilled migrants positively impact FDI entering their host country, suggesting a relation of complementarity between capital and skilled labor flows. More precisely, studying the influence of migration on inward FDI between 28 OECD countries and 162 partner countries for the year 2000, Dolman (2008) finds a positive impact of immigrants (in particular skilled immigrants) in OECD countries on inward FDI. Using a sample of 114 countries during the period 1990-2000, Docquier and Lodigiani (2010) find that inward FDI is positively related to skilled immigration. Foad (2012) finds similar results analyzing FDI and immigration from 10 source countries to the 50 US states, between 1991 and 2004. Immigrant communities in the USA attract FDI from their origin countries, this effect being stronger for migrants with a high education level. Some papers also finds that immigration fosters outward FDI. Analyzing US immigration and US FDI stocks into 56 partner countries for the years 1990 and 2000, Javorcik et al. (2011) find that immigration, in particular skilled immigration, leads outward FDI by providing native firms with better information on foreign countries, and by ensuring stronger contract enforcement. De Simone and Manchin (2012) find a positive effect between migration from Eastern to Western European countries and FDI in the reverse direction during the period 1995-2007.

Second, unskilled migrants negatively impact FDI entering their host country, suggesting a relation of substitution between capital and unskilled labor flows. More precisely, in their study of European countries over the period 1990-2000, El Yaman et al. (2007) find a contemporaneous substitution between unskilled immigration and FDI outflows and a dynamic complementarity between skilled immigration and FDI outflows toward the origin country of the migrants. Analyzing US immigration and FDI outflows in 1990 and 2000, Kugler and Rapoport (2007) corroborate these results. They find a contemporaneous substitution between low skilled migration and FDI, and a dynamic complementarity between high skilled migration and FDI.

Third, FDI from developed toward developing countries decrease reverse unskilled migration on the long run, suggesting a relation of substitution between capital and unskilled labor flows. More precisely, studying the link between FDI inflows from OECD to developing countries and migration in the reverse direction over the period 1991-2001, D'Agosto et al. (2006) find both a substitution and a complementarity effect. First, FDI positively impact the human capital accumulation of the developing country, and then increase wages which *in fine* decreases emigration. Second, FDI increase emigration by relaxing the budget constraint of would-be migrants. Conversely, Sanderson and Kentor (2008) find a positive relation. Working on 25 less-developed countries over the period 1985-2000, they show that FDI stocks positively impact emigration rates in the long run. Sanderson and Kentor (2008) corroborate the fact that FDI increases emigration, whereas Aroca and Maloney (2005) and Sauvant et al. (1993) corroborate the fact that FDI decreases emigration.

The usual explanations for these results come from outside the literature on international trade. The usual one focuses on the effect of migrant networks on trade and investment costs. De Simone and Manchin (2012) extend the 2x2x2 model of fragmentation and multinational production of Venables (1999) and show that, when immigrants relax the informational constraint faced by firms, immigration increases foreign investment. Federici and Giannetti (2010) develop a continuous time dynamic model. They consider a small open developing economy which lacks capital and skilled labor. They assume that migration is temporary, and that the capital stock in the developing country is generated by capital inflows from the developed countries hosting migrants. They show that return migration increases the human capital stock in the country and also act as an information revealing network. Thus, they attract inward FDI. Aubry et al. (2012) extend the theoretical framework of heterogeneous firms developed by Helpman et al. (2004, 2008), in which they introduce the idea that migrants reduce fixed costs supported by firms exporting and setting up horizontal FDI. Under the assumption that natives and foreigners are imperfect substitutes, they show that migration fosters trade and investments at the extensive and the intensive margins. In the same line, Wang (2013) extends the model of Helpman et al. (2004). He considers both low and high skilled workers. Under the assumption that natives and foreigners are imperfect substitutes, he shows that migrants allow for productivity gains as they bring foreign opportunities and thus increase the local competition.

Almost no attempt has been made to show how standard economic forces that are behind international trade may lead to complementarity of labor and capital flows. Ivlevs and De Melo (2010) use the 3x2 specific-factor framework, in which capital is mobile across sectors and labor is sector specific. In particular, they consider a non-traded good sector employing capital and unskilled labor, and an exported good sector employing capital and skilled labor. Under the assumption that capital is internationally mobile but labor is subject to national policy restrictions, they study the effects of exogenous high and low skilled migration shocks on FDI. Within this Ricardian framework, they find that skilled migration and FDI are complements. In particular, when exports are intensive in low skilled labor, emigration of high skilled labor leads to a positive capital outflow. In the same vein, Davis and Weinstein (2002) develop a Ricardian framework with a composite production factor. In this model, factor flows are motivated by the technological superiority of a country. Thus, skilled labor, unskilled labor and capital have a simultaneous incentive to enter the country with the highest technology.

Then, to the best of our knowledge, there is no model explaining the endogenous linkages between international factor flows and making the distinction between skilled and unskilled labor. Our paper tries to fill this gap using a general equilibrium model resting upon the standard

Heckscher-Ohlin (1919, 1933) framework, in which differences in factor endowments between regions are sufficient to explain factor flows. In this type of model, factor scarcities, such as capital deprivation, allow for corner solutions adequate to highlight substitution and complementarity between factor flows. We consider three internationally mobile factors (capital, unskilled and skilled labor) and two transportable goods, both internationally traded. The first good is a traditional one produced from labor only, skilled and unskilled workers being perfect substitutes; the production of the second good combines capital and labor, skilled and unskilled workers being perfect complements. We assume a Southern or developing economy, amply endowed with unskilled labor and poorly endowed with skilled labor; and a Northern or developed country, poorly endowed with unskilled labor and well endowed with skilled labor.

We start examining what happens when the whole capital stock, which is owned by the North, is also invested in the North. We then look at the changes induced by an exogenous transfer of capital from the North to the South. We obtain three possible equilibria: (i) an equilibrium when a small share of capital is invested in the South, (ii) an equilibrium when a large share of capital is invested in the South assuming that capital is worldly abundant, and (iii) an equilibrium when a large share of capital is invested in the South assuming that capital is worldly scarce. Then, we introduce imperfect international mobility of factors: investors ask for a risk premium for moving capital to the North and migrant workers must cover a migration cost. Looking at flows generated by changes in the risk premium and migration costs, we find a relation of substitution between capital and unskilled labor flows, and a relation of complementarity between capital and skilled labor flows.

The following section introduces the two-country model. Section 3 analyses the impact of exogenous capital transfers between countries. In section 4, we endogenise factor flows between countries and analyze the effects of factor transfers. Section 5 concludes, highlighting our results regarding the relation of substitution *versus* complementarity between factor flows, and suggesting some policy recommendations.

2 Model features

2.1 Single country features

Let us consider a single country, called South. The country combines three inputs, capital, skilled and unskilled labor, to produce two transportable goods, both internationally traded. The first good is the output of a traditional sector. The traditional sector does not use capital. Skilled and unskilled labor are perfect substitutes and returns to scale are constant. The production function of this sector is then $Q_1 = A(U_1 + cS_1)$, Q_1 being the output, U_1 and S_1 the respective inputs of unskilled and skilled labor, c a constant greater than unity, and A a positive constant. The intensive form of this production function is given by equation (1), with u_1 and s_1 being the technical coefficients *i.e.* the respective quantities of unskilled and skilled labor needed to produce one unit of output, respectively $u_1 = \frac{U_1}{Q_1}$ and $s_1 = \frac{S_1}{Q_1}$.

$$1 = A(u_1 + cs_1) \tag{1}$$

The second sector is a capitalist or industrial sector, as it employs capital in addition to labor. It is characterized by the following Cobb-Douglas function, $Q_2 = BK^{\beta} \left[min(U_2, S_2)\right]^{1-\beta}$, Q_2 being the output, K, U_2 and S_2 the respective inputs of capital, unskilled labor and skilled labor employed in the sector, β a constant between zero and unity, and B a positive constant. Then in the industrial sector, returns to scale are constant, capital and labor are imperfect substitutes to each other, and skilled and unskilled labor are perfect complements. The intensive form of the production function is given by equation (2), where k and l are the technical coefficients, respectively $k = \frac{K}{Q_2}$; $l = min(u_2, s_2)$, with $u_2 = \frac{U_2}{Q_2}$ and $s_2 = \frac{S_2}{Q_2}$.

$$1 = Bk^{\beta} l^{1-\beta} \tag{2}$$

Both outputs are perfectly mobile internationally, so that their prices are set up in international markets. We choose the traditional good as the numeraire, so that its price is unity, the price of the manufacturing good being p. We start assuming that the three factors are mobile between sectors of the economy but internationally immobile, so that their prices are determined locally. In perfectly competitive markets, the marginal productivity of factors equalize their prices. In the traditional sector, as long as both types of labor are employed in this sector, wages are given by equations (3) and (4), where w_u denotes the wage of unskilled labor, and w_s denotes the wage of skilled labor.

$$w_u = A \tag{3}$$

$$\frac{w_s}{c} = A \tag{4}$$

In the capitalist sector, factor prices are given by equations (5) and (6), where ρ denotes the returns to capital.

$$\rho k = \beta p \tag{5}$$

$$\left(w_u + w_s\right) l = (1 - \beta) p \tag{6}$$

Equilibrium implies the full employment of inputs. The total endowment of each factor equalizes the global demand by the two sectors of the economy:

$$U = u_1 Q_1 + l Q_2 \tag{7}$$

$$S = s_1 Q_1 + l Q_2 \tag{8}$$

$$K = k Q_2 \tag{9}$$

The solution to this system is developed in appendix, section A.1. For both types of labor to be employed in the traditional sector, the following inequality must hold:

$$K \left(Bl\right)^{1/\beta} = K \left[\frac{(1-\beta)\, Bp}{(1+c)A}\right]^{1/\beta} < \min\left(U, S\right) \tag{10}$$

What happens when this inequality does not hold depends upon the respective sizes of the skilled and unskilled labor forces. If there are less skilled workers than unskilled workers ($S < U$), there will be no skilled worker in the traditional sector, so that the skilled wage will no longer be given by equation (4). If there are more skilled workers than unskilled workers ($S > U$), there will be no unskilled worker in the traditional sector, so that the unskilled wage will no longer be given by equation (3). Both corner solutions are developed in appendix, section A.2.

2.2 The North/South framework

Let us now add a second country, called North. Apart their endowments in capital and both types of labor, both countries are similar to each other. Every variable x for the South will correspond to the variable x^* for the North. The world factor endowments are given by the sum of North and South endowments, such that $\bar{U} = U + U^*$; $\bar{S} = S + S^*$ and $\bar{K} = K + K^*$. We assume the South to be a developing economy, amply endowed with unskilled labor and poorly endowed with skilled labor. Conversely, the North is a developed country, poorly endowed with unskilled labor and well endowed with skilled labor. The global capital stock, $\bar{K}$, is fully owned by the North, but some part of this stock may be invested in the South.

In this two-country economy, factors are immobile and goods are perfectly mobile without any transaction cost. Then, there is a world market for each good and the local price equals the world price in both countries. As in the previous section, we normalize the price of the traditional good to unity, p being the price of the capitalist good.

In both countries, workers are endowed with preferences represented by a Cobb-Douglas utility function v, with $v(q_1, q_2) = q_1^{\gamma} q_2^{1-\gamma}$, where q_1 denotes the worker's consumption of the traditional good, q_2 denotes her consumption of the capitalist good, and γ is a constant between zero and unity ($\gamma \in]0;1[$). This utility function implies that every consumer devotes a share γ of her income for buying the traditional good and a share $1-\gamma$ for buying the capitalist good. Aggregating over all consumers and equalizing the world supply and the world demand, we get:

$$Q_1 + Q_1^* = \gamma \bar{I}$$
$$p\left(Q_2 + Q_2^*\right) = (1-\gamma)\bar{I}$$

where $\bar{I} = I + I^* = Q_1 + Q_1^* + p\left(Q_2 + Q_2^*\right)$ denotes the world income, $I = w_u U + w_s S + \rho K$ is the income generated in the South, and $I^* = w_u^* U^* + w_s^* S^* + \rho^* K^*$ is the income generated in the North. The Walras equality implies that we need to check one of these conditions only. The two conditions are equivalent to:

$$(1-\gamma)\left(Q_1 + Q_1^*\right) = \gamma p\left(Q_2 + Q_2^*\right) \tag{11}$$

It implies that at equilibrium, the part of the income generated by the traditional sector and devoted to buy the capitalist good, must equal the part of the income generated by the capitalist sector devoted to buy the traditional good.

3 Impact of an exogenous capital transfer

In this section, we start examining what happens when the whole capital stock, which is owned by the North, is also invested in the North. Then, we look at the changes induced by an exogenous transfer of capital from the North to the South.

3.1 Initial situation: the capital stock is fully invested in the North

At the initial stage, we assume the global stock of capital to be invested locally, so that the South has no capital ($K = 0$). As a result, in the South, the capitalist sector is inactive and all workers are hired by the traditional sector. In each sector, production equals:

$$Q_1 = A(U + cS)$$
$$Q_2 = 0$$

The wages of both types of workers are determined by the traditional sector only. Unskilled workers are paid $w_u = A$, while skilled workers are paid $w_s = cA$. Then, the global income earned by workers in the South equals:

$$I = (U + cS)\,A$$

In the North, the capital endowment $\left(K^* = \bar{K}\right)$ is large enough and unskilled labor is scarce enough for all the unskilled workers to be employed by the capitalist sector; the traditional sector employs skilled workers only. Then, the inequality:

$$U^* < \left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{1/\beta} \bar{K} < S^*$$

is met. In each sector, production equals:

$$Q_1^* = Ac\left(S^* - U^*\right)$$
$$Q_2^* = B\left(\bar{K}\right)^{\beta}\left(U^*\right)^{1-\beta}$$

Unskilled workers are paid $w_u^* = (1-\beta)Bp\left[\frac{\bar{K}}{U^*}\right]^{\beta} - cA$, while skilled workers are paid $w_s^* = cA$, and the returns to capital are $\rho^* = \beta Bp\left[\frac{U^*}{\bar{K}}\right]^{1-\beta}$. Then, the global income earned in the North equals:

$$I^* = Bp\left(\bar{K}\right)^{\beta}\left(U^*\right)^{1-\beta} + cA\left(S^* - U^*\right)$$

The world income equals:

$$\bar{I} = Bp\left(\bar{K}\right)^{\beta}\left(U^*\right)^{1-\beta} + A\left[U + c\left(\bar{S} - U^*\right)\right]$$

and then the equilibrium condition (11) gives the world price of the capitalist good:

$$p = \frac{1-\gamma}{\gamma}\frac{Q_1 + Q_1^*}{Q_2 + Q_2^*} = \frac{(1-\gamma)}{\gamma}\frac{A\left[U + c\left(\bar{S} - U^*\right)\right]}{B\left[\left(\bar{K}\right)^{\beta}\left(U^*\right)^{1-\beta}\right]}$$

At this stage, capital and unskilled labor have an incentive to move. In the North, returns to capital are $\rho^* = \beta Bp\left[\frac{U^*}{\bar{K}}\right]^{1-\beta}$. If a small quantity of capital were invested in the South, the returns to capital would be $\rho = \beta pk^{-1} = \beta Bp\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1}$. As long as the inequality $U^* < \left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{1/\beta} K < S^*$ holds, the returns to capital are higher in the South than in the North ($\rho > \rho^*$), and then there is an incentive for capital to move from the North to the South. The wage of unskilled workers in the North is higher than the wage in the South: $(1-\beta)Bp\left(\frac{K^*}{U^*}\right)^{\beta} - cA > A$. Then, if they were allowed to move with low enough migration costs, unskilled workers would move from the South to the North.

3.2 Types of capital transfers from the North to the South

Let us now look at what happens when the North transfers some part of its capital to the South, say $K > 0$, so that the capital endowment in the North is $K^* = \bar{K} - K$. The type of equilibrium the economy can reach after this transfer depends upon its impact on the constraints faced by the industrial sector in both countries. Let us remind that, in the initial situation, unskilled labor is scarce in the North while skilled labor is scarce in the South. There may be three main cases, depending upon the size of the capital transfer and the relative scarcities of skilled labor in the South and unskilled labor in the North.

The first case happens when the transfer is small enough for the capitalist sector in the South not to be able to employ all the skilled workers, so that some of them are still working in the traditional sector; and for the capitalist sector to be still large enough in the North for employing all the unskilled workers, so that the traditional sector employs skilled workers only. Then, the following constraints are met:

$$K < \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U, S) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S$$

$$\left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U^*, S^*) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^* < \bar{K} - K$$

so that:

$$K < \min\left(\left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S, \bar{K} - \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^*\right) \tag{12}$$

or equivalently:

$$\frac{U^*}{\bar{K} - K} < \left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{\frac{1}{\beta}} < \frac{S}{K}$$

$$\left(\frac{U^*}{\bar{K} - K}\right)^{1-\beta} < \left(\frac{(1-\beta)\,Bp}{(1+c)A}\right)^{\frac{1}{\beta}-1} < \left(\frac{S}{K}\right)^{1-\beta}$$

The second case occurs when the North transfers a large part of its capital to the South, while the global stock of capital is abundant. In that case, a large investment from the North to the

South leaves both countries with a large capital endowment. In both countries, the capitalist sector is important enough to drain all the scarce labor − skilled labor in the South and unskilled labor in the North − from the traditional sector, which only employs the abundant labor. The following inequalities hold:

$$K > \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U,S) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S$$

$$\bar{K} - K > \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U^*,S^*) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^*$$

so that:

$$\left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S < K < \bar{K} - \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^* \tag{13}$$

or equivalently:

$$\min\left(\frac{Q_2}{BK}\ ,\ \frac{Q_2^*}{B\left(\bar{K}-K\right)}\right) < \left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$

The third case happens when the North transfers a large part of its capital to the South, while the global stock of capital is relatively small. In that case, a large investment of capital from the North to the South leaves each country with a small endowment of capital. In both countries, the capitalist sector is not important enough to be able to employ all the scarce labor, so that this labor is still employed by the traditional sector. Then, the following inequalities hold:

$$K < \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U,S) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S$$

$$\bar{K} - K < \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} \min(U^*,S^*) = \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^*$$

or equivalently:

$$\bar{K} - \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} U^* < K < \left[\frac{(1+c)A}{(1-\beta)\,Bp}\right]^{\frac{1}{\beta}} S \tag{14}$$

3.3 Small exogenous investment to the South

Let us look at what happens when the North transfers a small part of its capital to the South, so that the condition (12) is met.

In the South, production in each sector equals:

$$Q_1 = A\left[U + cS - (1+c)\,K\left(\frac{B\,(1-\beta)\,p}{(1+c)A}\right)^{\frac{1}{\beta}}\right]$$

$$Q_2 = KB^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$

Because the traditional sector still employs both skilled and unskilled workers, both wages are determined by the traditional sector only, so that $w_u = A$ and $w_s = cA$. The returns to capital are $\rho = \beta (Bp)^{\frac{1}{\beta}} \left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1}$. Then, the global income generated in the South equals:

$$I = (U + cS) A + \beta (Bp)^{\frac{1}{\beta}} \left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1} K$$

In the North, productions equal:

$$Q_1^* = Ac (S^* - U^*)$$
$$Q_2^* = B \left(\bar{K} - K\right)^{\beta} (U^*)^{1-\beta}$$

Note that, using the expressions of Q_2 and Q_2^*, the small investment condition (12) may be written as:

$$\frac{Q_2^*}{\bar{K} - K} < \frac{Q_2}{K} < B \left(\frac{S}{K}\right)^{1-\beta} \tag{15}$$

The condition (15) holds as long as the average productivity of capital is higher in the South than in the North. Returns being constant, the same inequality holds for marginal productivity and returns.

Unskilled workers are paid $w_u^* = (1-\beta)Bp \left[\frac{\bar{K}-K}{U^*}\right]^{\beta} - cA$, while skilled workers are paid $w_s^* = cA$. The returns to capital are $\rho^* = \beta Bp \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}$. Then, the global income generated in the North equals:

$$I^* = cA (S^* - U^*) + Bp \left(\bar{K} - K\right)^{\beta} (U^*)^{1-\beta}$$

The world income equals:

$$\bar{I} = A \left[U + c \left(\bar{S} - U^*\right)\right] + \beta (Bp)^{\frac{1}{\beta}} \left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1} K + Bp \left(\bar{K} - K\right)^{\beta} (U^*)^{1-\beta}$$

Noting that $Q_1 = A (U + cS) - (1-\beta) pQ_2$, the equilibrium condition (11) becomes:

$$\begin{aligned} (1-\gamma) \left[A (U + cS) + Q_1^*\right] &= (1 - \beta + \beta\gamma) pQ_2 + \gamma p Q_2^* \qquad (16) \\ \Leftrightarrow (1-\gamma) A \left[U + c \left(\bar{S} - U^*\right)\right] &= (1 - \beta + \beta\gamma) (Bp)^{\frac{1}{\beta}} \left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1} K \\ &\quad + \gamma Bp \left(\bar{K} - K\right)^{\beta} (U^*)^{1-\beta} \end{aligned}$$

This equation has no explicit solution. However, as the left hand side is a positive constant and the right hand side increases from zero to infinity when p increases from zero to infinity, there is always an equilibrium price and this price is unique.

The derivatives of prices and quantities with respect to K are calculated in appendix, section A.3. A marginal increase in the amount of capital transferred to the South decreases the industrial production in the North ($dQ_2^*/dK < 0$) and increases it in the South ($dQ_2/dK > 0$). However, as long as the small investment condition holds (12), the industrial sector faces a shortage of unskilled labor in the North but not in the South. Then, the industrial production is more efficient in the

South; the increase in the South is larger than the decrease in the North ($dQ_2/dK + dQ_2^*/dK > 0$), and the global production of the industrial good increases. The industrial good is relatively more abundant so that its price decreases ($dp/dK < 0$). The traditional production does not change in the North ($dQ_1^*/dK = 0$) as the allocation of the labor force does not change; while it decreases in the South ($dQ_1/dK < 0$) as the capital transferred attracts workers previously working in the traditional sector. Then, the global production of the traditional good decreases. With a lower stock of capital in the North, the industrial sector is less constrained by the scarcity of unskilled labor ($d\rho^*/dK > 0$). At the same time, the wages in the South do not change as some of the scarce labor still works in the traditional sector. The skilled wage in the North does not change, as the sectoral allocation of labor remains unchanged. As for unskilled workers in the North, their wage decreases ($dw_u^*/dK < 0$), the departure of capital lowering their ability to extract a scarcity rent.

3.3.1 Small exogenous transfer of unskilled workers from the South to the North

An alternative strategy is to transfer a small amount of unskilled workers from the South to the North, capital being immobile ($K = 0$). As this is equivalent to a transfer of capital, this alternative factor allocation should identically impact global outputs and prices.

If $dU^* > 0$ unskilled migrants move from the South to the North, the unskilled labor force decreases in the South from U to $U - dU = U - dU^*$; in the North, it increases from U^* to $U^* + dU^*$. A small transfer of unskilled labor implies that the following condition is met:

$$U^* + dU^* < \left[\frac{(1-\beta)\, Bp}{(1+c)A}\right]^{\frac{1}{\beta}} \bar{K} < S^*$$

so that the unskilled labor is still abundant in the South and scarce in the North. Thus in the North, unskilled labor is still scarce enough compared to capital to be fully employed by the industrial sector.

Because the whole capital is still located in the North, unskilled labor is still scarce in the North and skilled labor is still scarce in the South, the equilibrium is still given by section 3.1. The impact of a transfer of unskilled workers may be found differentiating prices, quantities and factors returns with respect to U^*. These derivatives are presented in appendix, section A.4. A marginal transfer of unskilled labor decreases the production of the traditional sector in the South ($dQ_1/dU^* = -dQ_1/dU < 0$), to the benefit of the industrial sector in the North ($dQ_2^*/dU^* > 0$), which is now able to use more labor. But in the North, using more unskilled labor implies using more skilled labor, coming from the traditional sector where the output drops ($dQ_1^*/dU^* < 0$). As the production of the industrial sector increases, the world price of the manufactured good decreases ($dp/dU^* < 0$). A larger stock of unskilled labor in the North lessens the unskilled workers ability to extract a scarcity rent, so that their wage decreases ($dw_u^*/dU^* < 0$). The returns to capital decrease ($d\rho^*/dU^* < 0$) as the industrial sector is less constrained by the scarcity of unskilled labor.

Then, compared to a transfer of capital, migration of unskilled workers has the same global effects: in both cases, global production increases in the capitalist sector and decreases in the traditional sector, leading to a lower price of the manufactured good. The difference comes from the location of production. With a North-South capital transfer, the increase in global production of the industrial sector is located in the South, while production in the North decreases. Conversely, with a South-North migration of unskilled workers, the increase in global production of the industrial sector is located in the North, this industry being still absent in the South.

3.4 Large exogenous investment to the South when capital is worldly abundant

We now look at what happens when the North transfers a large part of its capital to the South, while the global stock of capital is abundant, so that the condition (13) is met.

The outputs of each sector in the South equal:

$$Q_1 = A\,(U - S)$$
$$Q_2 = BK^{\beta} S^{1-\beta}$$

Skilled workers are employed by the industrial sector only. The wage of skilled workers is $w_s = (1-\beta)Bp\left(\frac{K}{S}\right)^{\beta} - A$, the wage of unskilled workers is $w_u = A$, the returns to capital are $\rho = \beta Bp\left(\frac{S}{K}\right)^{1-\beta}$. The income generated in the South equals:

$$I = A\,(U - S) + BpK^{\beta} S^{1-\beta}$$

In the North, sectoral outputs are given by:

$$Q_1^* = Ac\,(S^* - U^*)$$
$$Q_2^* = B\left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}$$

The wages are $w_s^* = cA$ and $w_u^* = (1-\beta)Bp\left(\frac{\bar{K}-K}{U^*}\right)^{\beta} - cA$, the returns to capital are $\rho^* = \beta Bp\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}$. The income generated in the North equals:

$$I^* = cA\,(S^* - U^*) + Bp\left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}$$

The world income equals:

$$\bar{I} = A\,(U - S) + BpK^{\beta} S^{1-\beta} + cA\,(S^* - U^*) + Bp\left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}$$

The equilibrium condition (11) has an explicit solution, the world price of the industrial good being:

$$p = \frac{1-\gamma}{\gamma}\frac{Q_1 + Q_1^*}{Q_2 + Q_2^*} = \frac{1-\gamma}{\gamma}\frac{A\,(U - S + cS^* - cU^*)}{B\left[K^{\beta} S^{1-\beta} + \left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}\right]}$$

Introducing the expression of the price in condition (13), we find a condition on the capital endowments in each country:

$$\left[\phi\left(\frac{\widehat{Q}}{U^*} - 1\right)\left(\frac{U^*}{S}\right)^{1-\beta}\right]^{-1/\beta} < \frac{\bar{K}}{K} - 1 < \left[\phi\left(\frac{\widehat{Q}}{S} - 1\right)\left(\frac{S}{U^*}\right)^{1-\beta}\right]^{1/\beta} \tag{17}$$

where $\phi = \frac{(1-\beta)(1-\gamma)}{(1+c)\gamma}$ and $\widehat{Q} = U - S + c\,(S^* - U^*)$.

Derivatives with respect to K are presented in appendix, section A.5. When capital is worldly abundant, a marginal increase in the large amount of capital transferred from the North to the South increases the production of the industrial sector in the South ($dQ_2/dK > 0$) to the detriment of the North ($dQ_2^*/dK < 0$). We know that, compared to the local capital stock, unskilled labor is more scarce in the North than skilled labor in the South: $\frac{U^*}{\bar{K}-K} < \frac{S}{K}$. Thus, the industrial sector is less constrained by scarce labor in the South than in the North. As a result, the industrial production is more efficient in the South and the increase in production in the South is larger than the decrease in the North. Then, the world industrial output increases ($dQ_2/dK + dQ_2^*/dK > 0$), the production of the traditional sector in both countries remaining unchanged ($dQ_1/dK = dQ_1^*/dK = 0$). The industrial good becomes more abundant so that its price drops ($dp/dK < 0$). The wages of the abundant workers – unskilled workers in the South and skilled workers in the North – are not affected by a marginal transfer of capital. The wage of skilled workers in the South rises ($dw_s/dK > 0$) as there is more demand by the capitalist sector, so that they are in a better position to extract a scarcity rent. The wage of unskilled workers in the North drops ($dw_u^*/dK < 0$) as the outflow of capital reduces their ability to extract a scarcity rent. As capital flows to the South, its returns drop there ($d\rho/dK < 0$).

3.5 Large exogenous investment to the South when capital is worldly scarce

We finally look at what happens when the North transfers a large part of its capital to the South, the global stock of capital being relatively small, so that the condition (14) is met.

In the South, outputs in each sector are given by:

$$Q_1 = A\left[U + cS - (1+c)\,K\left(\frac{B\,(1-\beta)\,p}{A(1+c)}\right)^{\frac{1}{\beta}}\right]$$
$$Q_2 = KB^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$

while in the North, they are:

$$Q_1^* = A\left[U^* + cS^* - (1+c)\left(\bar{K} - K\right)\left(\frac{B\,(1-\beta)\,p}{A(1+c)}\right)^{\frac{1}{\beta}}\right]$$
$$Q_2^* = \left(\bar{K} - K\right)B^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$

so that the world production is:

$$Q_1 + Q_1^* = A\left[\bar{U} + c\bar{S} - (1+c)\,\bar{K}\left(\frac{B\,(1-\beta)\,p}{A(1+c)}\right)^{\frac{1}{\beta}}\right]$$
$$= A(\bar{U} + c\bar{S}) - (1-\beta)\,p\,(Q_2 + Q_2^*)$$
$$Q_2 + Q_2^* = \bar{K}B^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$

Then, the equilibrium condition (11) can be written:

$$p\,(Q_2 + Q_2^*) = \frac{1-\gamma}{\gamma}\,(Q_1 + Q_1^*) = \frac{1-\gamma}{\gamma}\left(A(\bar{U} + c\bar{S}) - (1-\beta)\,p\,(Q_2 + Q_2^*)\right)$$

so that:

$$p\,(Q_2 + Q_2^*) = \bar{K}\,(Bp)^{\frac{1}{\beta}}\left[\frac{(1-\beta)}{(1+c)A}\right]^{\frac{1}{\beta}-1} = \left(\frac{1-\gamma}{1-\beta+\gamma\beta}\right)A(\bar{U} + c\bar{S})$$

hence the equilibrium value of the price is:

$$p = \frac{A}{B}\left[\frac{1-\beta}{1+c}\right]^{\beta-1}\left[\frac{1-\gamma}{1-\beta+\gamma\beta}\right]^{\beta}\left(\frac{\bar{U} + c\bar{S}}{\bar{K}}\right)^{\beta}$$

In both countries, both types of workers are employed in both sectors. Then, the wages are $w_u = w_u^* = A$ and $w_s = w_s^* = cA$, and the returns to capital are $\rho = \rho^* = \beta\,(Bp)^{\frac{1}{\beta}}\left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1}$.

As long as the transfer of capital form the North to the South is large enough for Northern unskilled labor to be employed in both sectors, the price of the manufacturing good does not depend upon the allocation of capital. Therefore, a marginal change in the allocation of capital has no impact on the price ($dp/dK = 0$), the world production ($dQ_1/dK + dQ_1^*/dK = 0$ and $dQ_2/dK + dQ_2^*/dK = 0$), and factor returns. Only the international allocation of production changes: a higher amount of capital invested in the South implies a transfer of production of the manufacturing good from the North to the South ($dQ_2/dK = -dQ_2^*/dK > 0$), with the opposite effect for the traditional good ($dQ_1/dK = -dQ_1^*/dK < 0$).

4 International factors mobility

In this section, we look at what happens when factors are imperfectly mobile internationally. For accepting to invest in the South, capitalists must get a higher return than in the North, which may be interpreted as an exogenous risk premium, $\tau > 0$. Then:

$$K > 0 \Rightarrow \rho = \tau + \rho^* \tag{18}$$

Workers are internationally mobile. The initial populations of skilled and unskilled labor in the South and in the North are given by S_0, U_0, S_0^* and U_0^*. After migration, populations are given by S , U, S^* and U^*. Remember that the North is well endowed with skilled labor but not with unskilled labor, and the South is well endowed with unskilled labor but not with skilled labor. Therefore, we will be interested in movements of skilled workers from the North to the South, and of unskilled workers from the South to the North. As there are migration costs, for skilled workers to accept migrating from the North to the South, they must get a higher wage in the South, the difference needed for covering migration costs being $\mu_s > 0$. Then:

$$S > S_0 \;\Rightarrow\; w_s = \mu_s + w_s^* \tag{19}$$

Similarly, for unskilled workers to accept migrating from the South to the North, they must get a higher wage in the North, the difference needed for covering migration costs being $\mu_u > 0$. Then:

$$U^* > U_0^* \;\Rightarrow\; w_u^* = \mu_u + w_u \tag{20}$$

We study factors mobility for two equilibria. Each one results from a specific capital allocation between the two countries. The first equilibrium reflects the case in which a small share of the capital stock is invested in the South[1]. The second equilibrium corresponds to a situation in which capital is worldly abundant and invested in both countries. Note that when capital is largely invested in both countries but worldly scarce, case initially developed in section 3.5, factor returns are the same in both countries, then factors have no incentive to move.

4.1 Small investment to the South

4.1.1 The equilibrium

We look at what happens when, at equilibrium, a small share of the capital stock is invested in the South. Because of the small investment condition (16), the North is still abundant in capital so its industrial sector is compelled by unskilled labor. In the South, the industrial sector is compelled by capital and skilled labor. Equilibrium productions and factor returns have been determined hereinabove, in section 3.3. The allocations of capital, skilled and unskilled labor are now endogenous.

The capital returns in the North and the South are respectively $\rho^* = \beta Bp\left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}$ and $\rho = \beta\left(Bp\right)^{\frac{1}{\beta}}\left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1}$. Let us write $p\left(K, U^*\right)$ for the price of the manufacturing good when the quantity of capital invested in the North is K and there are U^* unskilled workers employed in the North. For capital to be invested in the North, the risk premium must be low enough:

$$\tau < \rho - \rho^* = \beta Bp\left(0, U_0^*\right)\left(\left[\frac{(1-\beta)\,Bp\left(0, U_0^*\right)}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U_0^*}{\bar{K}}\right]^{1-\beta}\right)$$

where the right hand side is the difference in returns to capital when the whole capital stock stays in the North and there is no migration of workers. If this equality is not met, there is no incentive for movement of capital. If it is met, some capital flows from the North to the South. At equilibrium, the condition (18) is met, which may be written as:

$$\tau = \rho - \rho^* = \beta Bp\left(K, U^*\right)\left(\left[\frac{(1-\beta)\,Bp\left(K, U^*\right)}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}\right)$$

[1]This equilibrium gives the same results than the equilibrium in which the capital is fully invested in the North ($K = 0$ and $K^* > 0$).

so that:

$$K = \bar{K} - U^* \left[\left[\frac{(1+c)A}{(1-\beta)\, Bp\,(K, U^*)} \right]^{\frac{1}{\beta}} - \left(\frac{\beta Bp\,(K, U^*)}{\tau} \right)^{\frac{1}{1-\beta}} \right]$$

The wages of skilled workers in the South and in the North are $w_s = w_s^* = cA$. They are identical in both countries and then skilled workers have no incentive to move: $S = S_0$ and $S^* = S_0^*$. The wages of unskilled workers in the South and in the North are $w_u = A$ and $w_u^* = (1-\beta)Bp\,(K, U^*) \left[\frac{\bar{K}-K}{U^*}\right]^{\beta} - cA$. For unskilled workers to move, the migration cost must be low enough:

$$\mu_u < w_u^* - w_u = (1-\beta)Bp\,(0, U_0^*) \left[\frac{\bar{K}}{U_0^*}\right]^{\beta} - (1+c)\,A$$

where the right hand side is the wage differential when the whole capital stock stays in the North and there is no migration of workers. If this equality is not met, there is no incentive for migration of unskilled workers. If it is met, some unskilled workers migrate from the South to the North, and at equilibrium the condition (20) is met, so that:

$$\mu_u = w_u - w_u^* = (1-\beta)Bp\,(K, U^*) \left[\frac{\bar{K}-K}{U^*}\right]^{\beta} - (1+c)\,A$$

which leads to:

$$U^* = \left[\frac{(1-\beta)Bp\,(K, U^*)}{\mu_u + (1+c)\,A}\right]^{\frac{1}{\beta}} \left(\bar{K} - K\right)$$

4.1.2 Impact of a marginal increase in the risk premium and the unskilled migration cost (τ, μ_u) on the factors allocation between countries

At equilibrium, we look at what happens to the factors allocation between countries when τ or μ_u varies. Calculations are presented in appendix, section A.6 and lead to the following result:

	$\frac{d\tau}{\tau}$	$\frac{d\mu_u}{\mu_u+(1+c)A}$
$\frac{dK}{K}$	-	+
$\frac{dU^*}{U^*}$	+	-

Table 1: Impact of a variation of τ and μ_u on the factors allocation between countries (small investment case).

A marginal increase in the risk premium, τ, implies that, for accepting to invest in the South, capitalists must get a higher South-North differential in returns to capital ($\rho^* - \rho$ increases). Thus, more capital stays in the North, and the capital stock decreases in the South ($\frac{dK}{K} < 0$) to the benefit of the North. A larger stock of capital staying in the North implies a higher demand for labor by the industrial sector. Unskilled labor being scarce in the North, this higher demand increases the wages of unskilled workers in the North, attracting new unskilled workers from the South ($\frac{dU^*}{U^*} > 0$).

A marginal increase in the migration cost for unskilled workers, μ_u, implies that Southern unskilled workers have less incentives to go North. Unskilled labor is retained in the South, and the stock of unskilled labor increases in the South to the detriment of the North ($\frac{dU^*}{U^*} < 0$). In the North, the lower supply of unskilled Southern workers implies that unskilled workers must get a higher wage, depressing the returns to capital. Investing in the South becomes a more attractive option and then the capital stock increases in the South ($\frac{dK}{K} > 0$), to the detriment of the North.

Note that, when the industrial sector is large in the North and small in the South, changes in the capital and unskilled labor allocation between countries have no consequence on the allocation of skilled and unskilled labor across sectors. The wages of skilled workers remain unchanged ($w_s = w_s^* = cA$), thereby skilled workers have no incentive to migrate ($dS = dS^* = 0$).

4.2 Large investment to the South when capital is worldly abundant

4.2.1 The equilibrium

We now look at what happens when, at equilibrium, a large share of capital is invested in the South, capital being worldly abundant ($K > 0$ and $K^* > 0$). Because of the large investment condition when capital is worldly abundant (17), the North is still abundant in capital so that its industrial sector is compelled by unskilled labor. In the South, the industrial sector is compelled by skilled labor. Equilibrium productions and factor returns have been determined hereinabove, in section 3.4.

At an interior equilibrium, capital, skilled and unskilled labor have no incentive to move. Then, the equilibrium conditions (18), (19) and (20) are met. Using the results from section 3.4, these equilibrium conditions may be written as:

$$\tau = \rho - \rho^* = \beta B p \left[\left(\frac{S}{K} \right)^{1-\beta} - \left(\frac{U^*}{\bar{K} - K} \right)^{1-\beta} \right]$$

$$\mu_s = w_s - w_s^* = (1-\beta)\, Bp \left(\frac{K}{S} \right)^{\beta} - (1+c)\, A$$

$$\mu_u = w_u^* - w_u = (1-\beta) Bp \left(\frac{\bar{K} - K}{U^*} \right)^{\beta} - (1+c)\, A$$

where the world price of the industrial good is:

$$p = \frac{1-\gamma}{\gamma} \frac{Q_1 + Q_1^*}{Q_2 + Q_2^*} = \frac{1-\gamma}{\gamma} \frac{A\left(\bar{U} + c\bar{S} - (1+c)\,(U^* + S)\right)}{B\left[K^{\beta} S^{1-\beta} + \left(\bar{K} - K\right)^{\beta} (U^*)^{1-\beta}\right]}$$

These four equations determine the equilibrium values of K, S, U^* and p.

4.2.2 Impact of a marginal increase in the risk premium and the migration costs (τ, μ_s, μ_u) on the factors allocation between countries

At equilibrium, we look at what happens to the factors allocation between countries when τ, μ_s or μ_u varies. Calculations are presented in appendix, section A.7 and lead to the following result:

	$\frac{d\tau}{\tau}$	$\frac{d\mu_u}{\mu_u+(1+c)A}$
$\frac{dK}{K}$	-	+
$\frac{dS}{S}$	-	+
$\frac{dU^*}{U^*}$	+	-

Table 2: Impact of a variation of τ and μ_u on the factors allocation between countries (large investment case).

At equilibrium, a marginal increase in the risk premium, τ, implies that capitalists expect a higher return in the North than what they can get in the South: $\tau > \rho - \rho^*$. More capital stays in the North to the detriment of the South ($\frac{dK}{K} < 0$). The industrial sector faces a shortage of unskilled labor in the North, and a shortage of skilled labor in the South. Thus, when the capitalist sector gets larger in the North and smaller in the South, the demand for skilled labor decreases in the South, and the demand for unskilled labor increases in the North. Both unskilled and skilled labor stocks increase in the North to the detriment of the South ($\frac{dU^*}{U^*} > 0$ and $\frac{dS}{S} < 0$).

A marginal increase in the migration cost for unskilled workers, μ_u, implies that the cost to go North becomes higher for unskilled workers: $\mu_u > w_u^* - w_u$. The stock of unskilled labor increases in the South to the detriment of the North ($\frac{dU^*}{U^*} < 0$). As the industrial sector is compelled by unskilled labor in the North, the sector gets smaller when the stock of unskilled labor decreases, and the demand for capital drops. The industrial sector gets larger in the South, but it is compelled by skilled labor. Thus, we expect the demand for skilled labor to increase in the South. Both skilled labor and capital stocks increase in the South to the detriment of the North ($\frac{dS}{S} > 0$ and $\frac{dK}{K} > 0$).

A marginal increase in the migration cost for skilled workers, μ_s, implies that the cost to go South becomes higher for skilled workers: $\mu_s > w_s - w_s^*$. Although the signs of $\frac{dK}{K}$, $\frac{dS}{S}$ and $\frac{dU^*}{U^*}$ are undetermined, we expect the following results: The stock of skilled labor should decrease in the South to the benefit of the North. In the South, the industrial sector is compelled by skilled labor, so the sector should get smaller when the stock of skilled labor decreases, and the demand for capital should drop. In the North, the industrial sector is compelled by unskilled labor, so the demand for unskilled labor should increase. As a result, we expect both capital and unskilled labor stocks to increase in the North to the detriment of the South.

5 Conclusion

Our paper reconciles the standard trade theory and available empirical evidence on the link between FDI and international migration flows. We highlight a relation of substitution between capital and unskilled labor flows, and a relation of complementarity between capital and skilled labor flows. In particular, our theoretical analysis corroborates studies showing that skilled immigrants positively impact inward FDI, and that unskilled immigrants negatively impact outward FDI toward their origin country (El Yaman et al., 2007; Kugler and Rapoport, 2007). Our results also corroborate empirical studies showing that migrants strengthen bilateral economic relations between their home and host countries (Docquier and Lodigiani, 2010; Beine et al., 2011; Aubry et al., 2012). More generally, our paper completes the theoretical analysis of the migration-FDI nexus, so far resting upon the effect of migrant networks on FDI costs to explain how migration foster FDI. Our approach is quite unique in this literature and allows us to analyze the endogenous links existing between factor flows *i.e.* the effects of FDI on migration, and conversely, in a single model.

With this theoretical exercise, we show that factor flows are highly interdependent and therefore that FDI and migration policy regimes should be thought together. We derive the following policy recommendations. Developed economies protecting their industries, may want to progressively increase immigration of low skilled workers to allow their industries to remain internationally competitive. On the other hand, our results imply that developed countries could, to some extent, regulate the volume of their low skilled immigration, implementing pro-active investment policies toward the migrant's origin countries.

Developing economies attracting FDI with an objective of economic development, may want to consider immigration policies supporting the arrival of high skilled workers on the short run, to meet the skilled labor demand of capitalist firms. In addition, developing countries may want to invest in education to increase their stock of human capital in the long run. On the other hand, our results imply that those countries could limit their emigration of low skilled workers by attracting multinational enterprises.

Finally, our study does not consider the case in which native and foreign workers are imperfect substitutes for firms, though it may be relevant for policy recommendations. On that issue, we make reference to the innovative work of Peri and Sparber (2009) and Ottaviano et al. (2013).

References

Aroca, P. and W. F. Maloney, "Migration, Trade, and Foreign Direct Investment in Mexico," *The World Bank Economic Review*, 2005, *19* (3), 449–472.

Aubry, A., M. Kugler, and H. Rapoport, "Migration, FDI and the margins of trade," Mimeo 2012.

Beine, M., F. Docquier, and Ç. Özden, "Diasporas," *Journal of Development Economics*, 2011, *95* (1), 30–41.

D'Agosto, E., N. Solferino, and G. Tria, "The Migration and FDI Puzzle: Complements or Substitutes?," CEIS Research Paper, Tor Vergata University, CEIS 2006.

Davis, D. R. and D. E. Weinstein, "Technological superiority and the losses from migration," NBER Working Paper Series 8971, National Bureau of Economic Research May 2002.

De Simone, G. and M. Manchin, "Outward Migration and Inward FDI: Factor Mobility between Eastern and Western Europe," *Review of International Economics*, 2012, *20* (3), 600–615.

Docquier, F. and E. Lodigiani, "Skilled migration and business networks," *Open Economies Review*, 2010, *21* (4), 565–588.

Dolman, B., "Migration, trade and investment," Staff Working Paper, Productivity Commission, Canberra 2008.

El Yaman, S., M. Kugler, and H. Rapoport, "Migrations et investissements directs étrangers dans l'espace européen (UE-15)," *Revue Économique*, 2007, *58* (3), 725–733.

Federici, D. and M. Giannetti, "Temporary migration and foreign direct investment," *Open Economies Review*, 2010, *21* (2), 293–308.

Foad, H., "FDI and immigration: a regional analysis," *The Annals of Regional Science*, 2012, *49* (1), 237–259.

Heckscher, E. F., "The effect of foreign trade on the distribution of income," *Ekonomisk Tidskriff*, 1919, pp. 497–512. Translated as chapter 13 in American Economic Association, Readings in the Theory of International Trade, Philadelphia: Blakiston, 1949, 272–300.

Helpman, E., M. J. Melitz, and S. R. Yeaple, "Export versus FDI with heterogenous firms," *American Economic Review*, 2004, *94* (1), 300–316.

_ , M. Melitz, and Y. Rubinstein, "Estimating trade flows: Trading partners and trading volumes," *The Quarterly Journal of Economics*, 2008, *123* (2), 441–487.

Ivlevs, A. and J. De Melo, "FDI, the Brain Drain and Trade: Channels and Evidence," *Annals of Economics and Statistics / Annales d'Économie et de Statistique*, 2010, (97–98), 103–121.

Javorcik, B. S., Ç. Özden, M. Spatareanu, and C. Neagu, "Migrant networks and foreign direct investment," *Journal of Development Economics*, 2011, *94* (2), 231–241.

Kugler, M. and H. Rapoport, "International labor and capital flows: Complements or substitutes?," *Economics Letters*, 2007, *94* (2), 155–162.

Ohlin, B. G., *Interregional and international trade*, Harvard University Press, 1933.

Ottaviano, G. I. P., G. Peri, and G. C. Wright, "Immigration, Offshoring, and American Jobs," *American Economic Review*, 2013, *103* (5), 1925–1959.

Peri, G. and C. Sparber, "Task Specialization, Immigration, and Wages," *American Economic Journal: Applied Economics*, 2009, *1* (3), 135–169.

Sanderson, M. R. and J. Kentor, "Foreign Direct Investment and International Migration: A Cross-National Analysis of Less-Developed Countries, 1985-2000," *International Sociology*, 2008, *23* (4), 514–539.

Sauvant, K. P., P. Mallampally, and P. Economou, "Foreign direct investment and international migration," *Transnational Corporations*, 1993, *2* (1), 33–69.

Venables, A. J., "Fragmentation and multinational production," *European Economic Review*, 1999, *43* (4–6), 935–945.

Wang, C.-K., "Migration and Multinationals: On the Welfare Effects of Firm and Labor Mobility," Mimeo 2013.

A Appendix

A.1 Single country (interior solution)

We assume that both factors are employed in the traditional sector, so that $u_1 > 0$ and $s_1 > 0$. Combining equations (3), (4) and (6), we obtain:

$$l = \frac{(1-\beta)\, p}{(1+c)A} \tag{21}$$

Then, using equations (2), (5) and (9), we obtain:

$$\begin{aligned} k &= B^{-\frac{1}{\beta}} l^{1-\frac{1}{\beta}} \\ \rho &= \beta p k^{-1} = \beta B^{\frac{1}{\beta}} p l^{\frac{1}{\beta}-1} \\ Q_2 &= k^{-1} K = K B^{\frac{1}{\beta}} l^{\frac{1}{\beta}-1} \end{aligned}$$

where l is given by equation (21). The wages are given by equations (3) and (4). Combining (21) with the equilibrium conditions (7) and (8) we get:

$$\begin{aligned} U &= u_1 Q_1 + l Q_2 = u_1 Q_1 + K B^{\frac{1}{\beta}} l^{\frac{1}{\beta}} \\ S &= s_1 Q_1 + l Q_2 = s_1 Q_1 + K B^{\frac{1}{\beta}} l^{\frac{1}{\beta}} \end{aligned}$$

Using the production function (1):

$$U + cS = (u_1 + c s_1)\, Q_1 + (1+c)\, K\, (Bl)^{\frac{1}{\beta}} = A^{-1} Q_1 + (1+c)\, K\, (Bl)^{\frac{1}{\beta}}$$

hence:

$$Q_1 = A \left[U + cS - (1+c)\, K\, (Bl)^{\frac{1}{\beta}} \right]$$

and:

$$\begin{aligned} u_1 &= \frac{U - K\,(Bl)^{\frac{1}{\beta}}}{Q_1} = \frac{U - K\,(Bl)^{\frac{1}{\beta}}}{A\left[U + cS - (1+c)\, K\, (Bl)^{\frac{1}{\beta}}\right]} \\ s_1 &= \frac{S - K\,(Bl)^{\frac{1}{\beta}}}{Q_1} = \frac{S - K\,(Bl)^{\frac{1}{\beta}}}{A\left[U + cS - (1+c)\, K\, (Bl)^{\frac{1}{\beta}}\right]} \end{aligned}$$

The conditions $u_1 > 0$ and $s_1 > 0$ imply:

$$K\,(Bl)^{\frac{1}{\beta}} = K \left[\frac{(1-\beta)\, Bp}{(1+c)A} \right]^{\frac{1}{\beta}} < \min\,(U, S)$$

A.2 Single country (corner solutions)

Let us first look at the case $u_1 > 0$ and $s_1 = 0$, which happens when:

$$S < K\,(Bl)^{\frac{1}{\beta}} = K \left[\frac{(1-\beta)\, Bp}{(1+c)A} \right]^{\frac{1}{\beta}} < U$$

Then, skilled labor is scarce enough for being employed in the capitalist sector only; the traditional sector employs unskilled labor only. If the wage of unskilled workers is still given by (3), the fact that skilled workers are no longer employed by the traditional sector implies that their wage is no longer given by (4). Knowing that $s_1 = 0$, and using equations (8) and (7), we get:

$$lQ_2 = S$$
$$u_1 Q_1 = U - lQ_2 = U - S$$

Then, using the production function (1) and the fact that $s_1 = 0$, we find:

$$Q_1 = A(u_1 + cs_1)Q_1 = Au_1Q_1 = A\left(U - S\right)$$

and combining equations (9), (2) and (8):

$$Q_2 = BK^\beta lQ_2^{1-\beta} = BK^\beta S^{1-\beta}$$
$$k = \frac{K}{Q_2} = B^{-1}\left(\frac{K}{S}\right)^{1-\beta}$$
$$l = \frac{S}{Q_2} = B^{-1}\left(\frac{S}{K}\right)^{\beta}$$

Last, using (3), (5) and (6), we get the wage of skilled workers and the returns to capital:

$$w_s = \frac{(1-\beta)\,p}{l} - w_u = (1-\beta)\,Bp\left(\frac{K}{S}\right)^{\beta} - A > cA$$
$$\rho = \beta p k^{-1} = \beta Bp\left(\frac{S}{K}\right)^{1-\beta}$$

Let us now look at the case $u_1 = 0$ and $s_1 > 0$, which happens when:

$$U < K\,(Bl)^{\frac{1}{\beta}} = K\left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{\frac{1}{\beta}} < S$$

Now, unskilled labor is scarce enough for being employed in the capitalist sector only; the traditional sector employs skilled labor only. The wage of skilled workers is still given by equation (4), but unskilled workers are no longer employed by the traditional sector, which implies that their wage is no longer given by (3). Knowing that $u_1 = 0$, and using (8) and (7), we get:

$$lQ_2 = U$$
$$s_1 Q_1 = S - lQ_2 = S - U$$

Then, using the production function (1) and the fact that $u_1 = 0$, we find:

$$Q_1 = A(u_1 + cs_1)Q_1 = Acs_1Q_1 = Ac\left(S - U\right)$$

and combining equations (9), (2) and (8):

$$Q_2 = BK^\beta lQ_2^{1-\beta} = BK^\beta U^{1-\beta}$$
$$k = \frac{K}{Q_2} = B^{-1}\left(\frac{K}{U}\right)^{1-\beta}$$
$$l = \frac{U}{Q_2} = B^{-1}\left(\frac{U}{K}\right)^{\beta}$$

Last, using (4) and (5), we get the wage of skilled workers and the returns to capital:

$$w_u = \frac{(1-\beta)\,p}{l} - w_s = (1-\beta)\,Bp\left(\frac{K}{U}\right)^{\beta} - cA > A$$
$$\rho = \beta p k^{-1} = \beta Bp\left(\frac{U}{K}\right)^{1-\beta}$$

A.3 Impact of a marginal transfer of capital on outputs and prices (small transfer case)

Let us look at the impact of a marginal transfer of capital from the North to the South on outputs and prices, when capital is abundant in the North and scarce in the South (case developed in section 3.3).

Using the equilibrium condition (16), we get:

$$(1-\gamma)\left[A\left(U+cS\right)+Q_1^*\right] = (1-\beta+\beta\gamma)\,pQ_2 + \gamma p Q_2^*$$
$$\Leftrightarrow (1-\gamma)\,A\left[U + c\left(\bar{S}-U^*\right)\right] = (1-\beta+\beta\gamma)\,(Bp)^{\frac{1}{\beta}}\left[\frac{1-\beta}{(1+c)A}\right]^{\frac{1}{\beta}-1} K + \gamma Bp\left(\bar{K}-K\right)^{\beta}\left(U^*\right)^{1-\beta}$$

Let us remind the outputs of each country:

$$Q_1 = A\left[U + cS - (1+c)\,K\left(\frac{B\,(1-\beta)\,p}{(1+c)A}\right)^{\frac{1}{\beta}}\right] = A\left(U+cS\right) - (1-\beta)\,pQ_2$$
$$Q_2 = KB^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1}$$
$$Q_1^* = Ac\left(S^*-U^*\right)$$
$$Q_2^* = B\left(\bar{K}-K\right)^{\beta}\left(U^*\right)^{1-\beta}$$

Differentiating the equilibrium condition (16), we get:

$$0 = \left[(1-\beta+\beta\gamma)\,\frac{\partial\left(pQ_2\right)}{\partial p} + \gamma\frac{\partial\left(pQ_2^*\right)}{\partial p}\right]dp + \left[(1-\beta+\beta\gamma)\,p\frac{\partial Q_2}{\partial K} + \gamma p\frac{\partial Q_2^*}{\partial K}\right]dK$$

and then, using the expressions of the sectoral outputs:

$$0 = \left[(1-\beta+\beta\gamma)\,\frac{1}{\beta}Q_2 + \gamma Q_2^*\right]dp + \left[(1-\beta+\beta\gamma)\,\frac{Q_2}{K} - \beta\gamma\frac{Q_2^*}{\bar{K}-K}\right]pdK$$

where:

$$\frac{\partial\left(pQ_2\right)}{\partial p} = \frac{1}{\beta}Q_2,\ \frac{\partial Q_2}{\partial K} = \frac{Q_2}{K}$$
$$\frac{\partial\left(pQ_2^*\right)}{\partial p} = Q_2^*,\ \frac{\partial Q_2^*}{\partial K} = -\beta\frac{Q_2^*}{\bar{K}-K}$$

Therefore, the derivative of the price of the industrial good with respect to capital is:

$$\frac{dp}{dK} = \frac{\beta p\left[\beta\gamma\frac{Q_2^*}{\bar{K}-K} - (1-\beta+\beta\gamma)\,\frac{Q_2}{K}\right]}{(1-\beta+\beta\gamma)\,Q_2 + \beta\gamma Q_2^*}$$

Now, using the small investment condition (15):

$$\frac{Q_2^*}{\bar{K}-K} < \frac{Q_2}{K}$$

we find that the price of the industrial good decreases with a small transfer of capital:

$$\frac{dp}{dK} = \frac{\beta p \left[\beta\gamma\left(\frac{Q_2^*}{\bar{K}-K} - \frac{Q_2}{K}\right) - (1-\beta)\frac{Q_2}{K}\right]}{(1-\beta+\beta\gamma)\,Q_2 + \beta\gamma Q_2^*} < 0$$

The impact of this transfer of capital on the sectoral outputs in each country, is given by:

$$\begin{aligned}
\frac{dQ_1}{dK} &= -(1-\beta)\frac{d(pQ_2)}{dK} < 0 \\
\frac{dQ_2}{dK} &> 0 \\
\frac{dQ_1^*}{dK} &= 0 \\
\frac{dQ_2^*}{dK} &= -\beta\frac{Q_2^*}{\bar{K}-K} < 0
\end{aligned}$$

the second equality coming from:

$$\begin{aligned}
\frac{dQ_2}{Q_2} &= \frac{dK}{K} + \left(\frac{1}{\beta}-1\right)\frac{dp}{p} \\
\frac{1}{Q_2}\frac{dQ_2}{dK} &= \frac{1}{K} + \left(\frac{1}{\beta}-1\right)\frac{1}{p}\frac{dp}{dK} \\
&= \frac{1}{K} + \frac{(1-\beta)\left[\beta\gamma\frac{Q_2^*}{\bar{K}-K} - (1-\beta+\beta\gamma)\frac{Q_2}{K}\right]}{(1-\beta+\beta\gamma)\,Q_2+\beta\gamma Q_2^*} \\
\frac{dQ_2}{dK} &= \frac{\left(\frac{1-\beta}{\bar{K}-K}+\frac{1}{K}\right)\beta\gamma Q_2^* + \beta(1-\beta+\beta\gamma)\frac{Q_2}{K}}{(1-\beta+\beta\gamma)\,Q_2+\beta\gamma Q_2^*}Q_2 > 0
\end{aligned}$$

while the first equality comes from:

$$\begin{aligned}
\frac{d(pQ_2)}{pQ_2} &= \frac{dK}{K} + \frac{1}{\beta}\frac{dp}{p} \\
\frac{1}{pQ_2}\frac{d(pQ_2)}{dK} &= \frac{1}{K} + \frac{1}{\beta p}\frac{dp}{dK} \\
&= \frac{1}{K} + \frac{\beta\gamma\frac{Q_2^*}{\bar{K}-K} - (1-\beta+\beta\gamma)\frac{Q_2}{K}}{(1-\beta+\beta\gamma)\,Q_2+\beta\gamma Q_2^*} \\
\frac{d(pQ_2)}{dK} &= \frac{\beta\gamma\left(\frac{Q_2^*}{\bar{K}-K}+\frac{Q_2^*}{K}\right)}{(1-\beta+\beta\gamma)\,Q_2+\beta\gamma Q_2^*}pQ_2 > 0
\end{aligned}$$

The transfer decreases the world output of the traditional sector:

$$\frac{d(Q_1+Q_1^*)}{dK} = \frac{dQ_1}{dK} < 0$$

and increases the world output of the industrial sector:

$$\begin{aligned}
\frac{d\left(Q_2+Q_2^*\right)}{dK} &= Q_2\frac{\left(\frac{1-\beta}{\bar{K}-K}+\frac{1}{K}\right)\beta\gamma Q_2^*+\beta\left(1-\beta+\beta\gamma\right)\frac{Q_2}{K}}{\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*}-\beta\frac{Q_2^*}{\bar{K}-K}\\
&=\beta\frac{\left(\frac{1-\beta}{\bar{K}-K}+\frac{1}{K}\right)\gamma Q_2Q_2^*+\left(1-\beta+\beta\gamma\right)Q_2\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\left[\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*\right]}{\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*}\\
&=\beta\frac{\left[\left(\frac{\bar{K}}{K}-\beta\right)Q_2-\beta Q_2^*\right]\gamma\frac{Q_2^*}{\bar{K}-K}+\left(1-\beta+\beta\gamma\right)\left(\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right)Q_2}{\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*}\\
&=\beta\frac{\left[\left(\bar{K}-\beta K\right)\frac{Q_2}{K}-\beta\left(\bar{K}-K\right)\frac{Q_2^*}{\bar{K}-K}\right]\gamma\frac{Q_2^*}{\bar{K}-K}+\left(1-\beta+\beta\gamma\right)\left(\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right)Q_2}{\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*}\\
&=\beta\frac{\left(1-\beta\right)\bar{K}\gamma\frac{Q_2}{K}\frac{Q_2^*}{\bar{K}-K}+\left(\beta\gamma Q_2^*+\left(1-\beta+\beta\gamma\right)Q_2\right)\left(\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right)}{\left(1-\beta+\beta\gamma\right)Q_2+\beta\gamma Q_2^*}>0
\end{aligned}$$

The wages in the South and the skilled wage in the North are unchanged. Only the unskilled wage in the North changes:

$$\begin{aligned}
w_u^* &= (1-\beta)Bp\left[\frac{\bar{K}-K}{U^*}\right]^\beta-cA=(1-\beta)p\frac{Q_2^*}{U^*}-cA\\
\frac{dw_u^*}{dK} &= \frac{(1-\beta)}{U^*}\frac{d(pQ_2^*)}{dK}=\frac{(1-\beta)}{U^*}\left(\frac{dp}{dK}Q_2^*+\frac{dQ_2^*}{dK}p\right)<0
\end{aligned}$$

The decrease in the production of the Northern capitalist sector generated by the capital transfer to the South decreases the ability of Northern unskilled workers to extract a scarcity rent.

A.4 Impact of a marginal transfer of unskilled labor on outputs and prices (small transfer case)

Let us look at the impact of a marginal transfer of unskilled labor from the South to the North on outputs and prices, when capital is fully invested in the North, and unskilled labor is transferred from the South to the North (case developed in section 3.3). Then, in the North, the labor force increases from U^* to U^*+dU^*, with $dU^*>0$, while in the South the labor force decreases from U to $U-dU$, with $-dU=-dU^*<0$.

Let us remind that, as long as unskilled labor is scarce in the North and abundant in the South, the outputs of each country and the price of the industrial good are given by:

$$\begin{aligned}
Q_1 &= A\left(U+cS\right)\\
Q_2 &= 0\\
Q_1^* &= Ac\left(S^*-U^*\right)\\
Q_2^* &= B\left(\bar{K}\right)^\beta\left(U^*\right)^{1-\beta}\\
p &= \frac{1-\gamma}{\gamma}\frac{Q_1+Q_1^*}{Q_2+Q_2^*}=\frac{(1-\gamma)}{\gamma}\frac{A\left[U+c\left(\bar{S}-U^*\right)\right]}{B\left(\bar{K}\right)^\beta\left(U^*\right)^{1-\beta}}
\end{aligned}$$

Using logarithmic derivatives, we get the derivative of the price of the industrial good with respect to a small transfer from South to North:

$$\frac{1}{p}\frac{dp}{dU^*} = \left[-\frac{dQ_1}{dU} + \frac{dQ_1^*}{dU^*}\right]\frac{1}{Q_1+Q_1^*} - \frac{1}{Q_2^*}\frac{dQ_2^*}{dU^*}$$
$$\frac{dp}{dU^*} = \left[-(1+c)\frac{1}{U+c\left(\bar{S}-U^*\right)} - (1-\beta)\frac{1}{U^*}\right]p < 0$$

The output changes are:

$$-\frac{dQ_1}{dU} = -A < 0$$
$$\frac{dQ_1^*}{dU^*} = -cA < 0$$
$$-\frac{dQ_2}{dU} = 0$$
$$\frac{dQ_2^*}{dU^*} = (1-\beta)\frac{Q_2^*}{U^*} > 0$$

and:

$$\frac{d\left(Q_1+Q_1^*\right)}{dU^*} = -(1+c)A < 0$$
$$\frac{d\left(Q_2+Q_2^*\right)}{dU^*} = (1-\beta)\frac{Q_2^*}{(U^*+M)} > 0$$

Let remind the expression of the wage of unskilled workers in the North:

$$w_u^* = (1-\beta)Bp\left[\frac{\bar{K}}{U^*}\right]^{\beta} - cA = \frac{1-\beta}{U^*}pQ_2^* - cA$$

Using logarithmic derivatives, we find that the wage of unskilled workers decreases in the North:

$$\frac{dw_u^*}{dU^*} = \frac{1-\beta}{U^*}\left[\frac{dp}{dU^*}Q_2^* + \frac{dQ_2^*}{dU^*}p\right] = \frac{1-\beta}{U^*}\left[\frac{d\left(pQ_2^*\right)}{dU^*}\right]$$
$$= \frac{1-\beta}{U^*}\left[-(1+c)\frac{B\left(\bar{K}\right)^{\beta}\left(U^*\right)^{1-\beta}}{U+c\left(\bar{S}-U^*\right)}\right]p$$
$$= -\frac{1-\beta}{U^*}\frac{(1-\gamma)}{\gamma}A(1+c) < 0$$

Let remind the expression of capital returns in the North:

$$\rho^* = \beta Bp\left[\frac{U^*}{\bar{K}}\right]^{1-\beta} = \beta p\frac{Q_2^*}{\bar{K}}$$

Using logarithmic derivatives we find that the returns decreases in the North:

$$\frac{d\rho^*}{dU^*} = \frac{\beta}{\bar{K}}\left[\frac{dp}{dU^*}Q_2^* + \frac{dQ_2^*}{dU^*}p\right]$$
$$= \frac{\beta}{\bar{K}}\left[-(1+c)\frac{B\left(\bar{K}\right)^{\beta}(U^*)^{1-\beta}}{U + c\left(\bar{S} - U^*\right)}\right]p$$
$$= -A(1+c)\frac{\beta}{\bar{K}}\frac{(1-\gamma)}{\gamma} < 0$$

A.5 Impact of a marginal transfer of capital on outputs and prices (large transfer case with abundant capital endowment)

Let us look at the impact of a marginal transfer of capital from the North to the South on outputs and prices, when the global capital stock is large enough for capital to become abundant in both countries (case developed in section 3.4).

Let us remind the expressions of the outputs in each country and the price of the industrial good:

$$Q_1 = A(U - S)$$
$$Q_2 = BK^{\beta}S^{1-\beta}$$
$$Q_1^* = Ac(S^* - U^*)$$
$$Q_2^* = B\left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}$$
$$p = \frac{1-\gamma}{\gamma}\frac{Q_1 + Q_1^*}{Q_2 + Q_2^*} = \frac{1-\gamma}{\gamma}\frac{A(U - S + cS^* - cU^*)}{B\left[K^{\beta}S^{1-\beta} + \left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}\right]}$$

Using logarithmic derivatives, we get the derivative of the price of the industrial good with respect to capital:

$$\frac{1}{p}\frac{dp}{dK} = -\beta\frac{\left(\frac{S}{K}\right)^{1-\beta} - \left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}{K^{\beta}S^{1-\beta} + \left(\bar{K} - K\right)^{\beta}(U^*)^{1-\beta}}$$
$$\frac{dp}{dK} = -\beta p\frac{\left[\frac{Q_2}{K} - \frac{Q_2^*}{\bar{K}-K}\right]}{Q_2 + Q_2^*} < 0$$

The impact of a marginal transfer of capital on outputs is:

$$\frac{dQ_1}{dK} = 0$$
$$\frac{dQ_1^*}{dK} = 0$$
$$\frac{dQ_2}{dK} = \beta\frac{Q_2}{K} = \beta B\left(\frac{S}{K}\right)^{1-\beta} > 0$$
$$\frac{dQ_2^*}{dK} = -\beta\frac{Q_2^*}{\bar{K} - K} = -\beta B\left(\frac{U^*}{\bar{K} - K}\right)^{1-\beta} < 0$$

Globally, the output of the traditional sector is not affected by a marginal transfer of capital:

$$\frac{d\left(Q_1+Q_1^*\right)}{dK}=0$$

Note that the feasibility condition (13) does not allow us to determine the sign of:

$$\frac{d\left(Q_2+Q_2^*\right)}{dK}=\beta B\left[\left(\frac{S}{K}\right)^{1-\beta}-\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}\right]$$

However, we know that the price of the industrial good decreases $\left(\frac{dp}{dK}<0\right)$, which implies that the good becomes more abundant. Thus we can deduct that the global input of the industrial good increases, so that unskilled labor is more scarce in the North than skilled labor is in the South. The industrial production is more efficient in the South, and the increase in production in the South is larger than the decrease in the North. Then, the world industrial output increases $\left(\frac{d(Q_2+Q_2^*)}{dK}>0\right)$.

The impact on the wage of skilled workers in the South is positive, as $w_s=(1-\beta)Bp\left(\frac{K}{S}\right)^{\beta}-A=(1-\beta)p\frac{Q_2}{S}-A$ implies:

$$\begin{aligned}
\frac{dw_s}{dK} &= \frac{(1-\beta)}{S}\left[\frac{dp}{dK}Q_2+\frac{dQ_2}{dK}p\right] \\
&= (1-\beta)B\left[\left(\frac{K}{S}\right)^{\beta}\frac{dp}{dK}+\beta\frac{p}{K}\left(\frac{K}{S}\right)^{\beta}\right] \\
&= (1-\beta)B\left(\frac{K}{S}\right)^{\beta}\left[\frac{dp}{dK}+\beta\frac{p}{K}\right] \\
&= (1-\beta)B\left(\frac{K}{S}\right)^{\beta}\beta\frac{p}{K}\left[1-\frac{\left(\frac{S}{K}\right)^{1-\beta}-\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}{\left(\frac{S}{K}\right)^{1-\beta}+\left(\frac{\bar{K}}{K}-1\right)\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}\right] \\
&= (1-\beta)B\left(\frac{K}{S}\right)^{\beta}\beta p\frac{\bar{K}\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}{\left(\frac{S}{K}\right)^{1-\beta}+\left(\frac{\bar{K}}{K}-1\right)\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}>0
\end{aligned}$$

The impact on the wage of unskilled workers in the North is negative, as $w_u^*=(1-\beta)Bp\left(\frac{\bar{K}-K}{U^*}\right)^{\beta}-cA=(1-\beta)p\frac{Q_2^*}{U^*}-cA$ implies:

$$
\begin{aligned}
\frac{dw_u^*}{dK} &= \frac{(1-\beta)}{U^*}\left[\frac{dp}{dK}Q_2^* + \frac{dQ_2^*}{dK}p\right] \\
&= (1-\beta)B\left(\frac{\bar{K}-K}{U^*}\right)^{\beta}\left[\frac{dp}{dK} - \beta\frac{p}{\bar{K}-K}\right] \\
&= -(1-\beta)B\left(\frac{\bar{K}-K}{U^*}\right)^{\beta}\frac{\beta p}{\bar{K}-K}\left[\frac{\left(\frac{S}{K}\right)^{1-\beta} - \left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}{\frac{K}{\bar{K}-K}\left(\frac{S}{K}\right)^{1-\beta} + \left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}} + 1\right] \\
&= -(1-\beta)B\left(\frac{\bar{K}-K}{U^*}\right)^{\beta}\beta p\frac{\bar{K}\left(\frac{S}{K}\right)^{1-\beta}}{\frac{K}{\bar{K}-K}\left(\frac{S}{K}\right)^{1-\beta} + \left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}} < 0
\end{aligned}
$$

Let us remind the expression of the returns in the South:

$$\rho = \beta B p\left(\frac{S}{K}\right)^{1-\beta}$$

Using logarithmic derivatives, we get the derivative of the returns to capital in the South with respect to a capital variation:

$$\frac{1}{\rho}\frac{d\rho}{dK} = \frac{1}{p}\frac{dp}{dK} - \frac{(1-\beta)}{K} < 0$$

the negative sign coming from the fact that $\frac{dp}{dK} < 0$.

In the North, returns to capital are:

$$\rho^* = \beta B p\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}$$

Using logarithmic derivatives, we get the derivative of the returns to capital in the North with respect to a capital variation:

$$\frac{1}{\rho^*}\frac{d\rho^*}{dK} = \frac{1}{p}\frac{dp}{dK} + \frac{(1-\beta)}{\bar{K}-K}$$

and, after straightforward calculations:

$$\frac{d\rho^*}{dK} = \frac{\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}\left[1 + \left((1-\beta)\frac{K}{\bar{K}-K} - \beta\right)\left(\frac{S}{U^*}\frac{\bar{K}-K}{K}\right)^{1-\beta}\right]}{K^{\beta}S^{1-\beta} + \left(\bar{K}-K\right)^{\beta}(U^*)^{1-\beta}}\rho^*$$

Unhappily, this derivative cannot be signed.

Notice that capital has no more incentive to flow from the North to the South when returns equalize in both countries, ($\rho^* = \rho$), which happens when:

$$K = \left(\frac{S}{U^*} + 1\right)\bar{K} \Leftrightarrow \frac{U^*}{\bar{K}-K} = \frac{S}{K}$$

In other words, at the equilibrium the relative scarcity of unskilled labor compared to the capital stock in the North equalizes the relative scarcity of skilled labor compared to the capital stock in the South. This capital allocation among countries maximizes the production of the industrial sector.

A.6 Impact of a risk premium and a migration cost variation on the factors allocation (small transfer case)

Let us look at the impact of a risk premium and an unskilled migration cost variations on the factors allocation among countries, for a small investment to the South (case developed in section 3.3).

Let us remind the equilibrium equations:

$$\tau = \beta Bp\left(\left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}\right) = \beta p\left(\frac{Q_2}{K} - \frac{Q_2^*}{\bar{K}-K}\right)$$

$$\mu_u = (1-\beta)pB\left[\frac{\bar{K}-K}{U^*}\right]^{\beta} - (1+c)\,A = (1-\beta)p\frac{Q_2^*}{U^*} - (1+c)\,A$$

The logarithmic derivatives of these equilibrium conditions are:

$$\begin{aligned}
\frac{d\tau}{\tau} &= \frac{dp}{p} + \frac{d\left(\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}\right)}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}} \\
&= \frac{dp}{p} + \frac{\left(\frac{1}{\beta}-1\right)\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1}\frac{dp}{p} - (1-\beta)\left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}\left(\frac{dU^*}{U^*} + \frac{dK}{\bar{K}-K}\right)}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}} \\
&= \left(1 + \frac{\left(\frac{1}{\beta}-1\right)\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1}}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}}\right)\frac{dp}{p} - \frac{(1-\beta)\left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}}\left(\frac{dU^*}{U^*} + \frac{dK}{\bar{K}-K}\right) \\
\frac{d\mu_u}{\mu_u + (1+c)\,A} &= \frac{dp}{p} - \beta\left(\frac{dK}{\bar{K}-K} + \frac{dU^*}{U^*}\right)
\end{aligned}$$

which may be written as:

$$\frac{d\tau}{\tau} = \left(1+\beta^{-1}D\right)\frac{dp}{p} - C\left(\frac{dU^*}{U^*} + \frac{K}{\bar{K}-K}\frac{dK}{K}\right)$$

$$\frac{d\mu_u}{\mu_u + (1+c)\,A} = \frac{dp}{p} - \beta\left(\frac{K}{\bar{K}-K}\frac{dK}{K} + \frac{dU^*}{U^*}\right)$$

where:

$$C = \frac{(1-\beta)\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}} = (1-\beta)\,\frac{\beta Bp}{\tau}\left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta} > 0$$

$$D = \frac{(1-\beta)\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1}}{\left[\frac{(1-\beta)Bp}{(1+c)A}\right]^{\frac{1}{\beta}-1} - \left[\frac{U^*}{\bar{K}-K}\right]^{1-\beta}} = (1-\beta)\,\frac{\beta Bp}{\tau}\left[\frac{(1-\beta)\,Bp}{(1+c)A}\right]^{\frac{1-\beta}{\beta}} > 0$$

$$D - C = 1-\beta$$

As there is no explicit solution for the price, we use the total derivative of the equilibrium condition:

$$(1-\gamma)\,A\left[U+c\left(\bar{S}-U^*\right)\right]=(1-\beta+\beta\gamma)\,pQ_2+\gamma pQ_2^*$$

with the logarithmic derivatives of the outputs:

$$\begin{aligned}
Q_2 &= KB^{\frac{1}{\beta}}\left[\frac{(1-\beta)\,p}{(1+c)A}\right]^{\frac{1}{\beta}-1} \Longrightarrow \frac{dQ_2}{Q_2}=\frac{dK}{K}+\left(\frac{1}{\beta}-1\right)\frac{dp}{p}\\
Q_2^* &= B\left(\bar{K}-K\right)^{\beta}\left(U^*\right)^{1-\beta} \Longrightarrow \frac{dQ_2^*}{Q_2^*}=-\beta\frac{K}{\bar{K}-K}\frac{dK}{K}+(1-\beta)\,\frac{dU^*}{U^*}
\end{aligned}$$

to get an expression of dp/p:

$$\begin{aligned}
-(1-\gamma)\,A\,(1+c)\,U^*\frac{dU^*}{U^*} &= (1-\beta+\beta\gamma)\,pQ_2\left(\frac{dp}{p}+\frac{dQ_2}{Q_2}\right)+\gamma pQ_2^*\left(\frac{dp}{p}+\frac{dQ_2^*}{Q_2^*}\right)\\
&= (1-\beta+\beta\gamma)\,pQ_2\left(\frac{dK}{K}+\frac{1}{\beta}\frac{dp}{p}\right)+\gamma pQ_2^*\left[\frac{dp}{p}-\beta\frac{K}{\bar{K}-K}\frac{dK}{K}+(1-\beta)\,\frac{dU^*}{U^*}\right]\\
&= \left(\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*\right)p\frac{dp}{p}+\left[(1-\beta+\beta\gamma)\,Q_2-\beta\gamma Q_2^*\frac{K}{\bar{K}-K}\right]p\frac{dK}{K}\\
&\quad+(1-\beta)\,\gamma pQ_2^*\frac{dU^*}{U^*}\\
\Leftrightarrow \frac{dp}{p} &= -\frac{(1-\beta+\beta\gamma)\,Q_2-\beta\gamma Q_2^*\frac{K}{\bar{K}-K}}{\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*}\frac{dK}{K}-\frac{(1-\beta)\,\gamma pQ_2^*+(1-\gamma)\,A\,(1+c)\,U^*}{\left(\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*\right)p}\frac{dU^*}{U^*}\\
&= -E_K\frac{dK}{K}-E_U\frac{dU^*}{U^*}
\end{aligned}$$

where:

$$\begin{aligned}
E_K &= \frac{(1-\beta+\beta\gamma)\,Q_2-\beta\gamma Q_2^*\frac{K}{\bar{K}-K}}{\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*}=\frac{(1-\beta)\,Q_2+\beta\gamma K\left(\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right)}{\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*}>0\\
E_U &= \frac{(1-\beta)\,\gamma pQ_2^*+(1-\gamma)\,A\,(1+c)\,U^*}{\left(\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*\right)p}>0
\end{aligned}$$

the positive sign of E_K coming from the small investment condition (15).

Then:

$$\begin{aligned}
\frac{d\tau}{\tau} &= -\left(1+\beta^{-1}D\right)\left(E_K\frac{dK}{K}+E_U\frac{dU^*}{U^*}\right)-C\left(\frac{dU^*}{U^*}+\frac{K}{\bar{K}-K}\frac{dK}{K}\right)\\
&= -\left[\left(1+\beta^{-1}D\right)E_K+C\frac{K}{\bar{K}-K}\right]\frac{dK}{K}-\left[\left(1+\beta^{-1}D\right)E_U+C\right]\frac{dU^*}{U^*}\\
\frac{d\mu_u}{\mu_u+(1+c)\,A} &= -E_K\frac{dK}{K}-E_U\frac{dU^*}{U^*}-\beta\left(\frac{K}{\bar{K}-K}\frac{dK}{K}+\frac{dU^*}{U^*}\right)\\
&= -\left(E_K+\frac{\beta K}{\bar{K}-K}\right)\frac{dK}{K}-(E_U+\beta)\,\frac{dU^*}{U^*}
\end{aligned}$$

which may be written as:

$$M\begin{bmatrix}\frac{dK}{K}\\ \frac{dU^*}{U^*}\end{bmatrix}=-\begin{bmatrix}\frac{d\tau}{\tau}\\ \frac{d\mu_u}{\mu_u+(1+c)A}\end{bmatrix}$$

where:

$$M=\left[\begin{array}{cc}\left(1+\beta^{-1}D\right)E_K+C\frac{K}{\bar{K}-K} & \left(1+\beta^{-1}D\right)E_U+C\\ E_K+\frac{\beta K}{\bar{K}-K} & E_U+\beta\end{array}\right]$$

so that:

$$\left[\begin{array}{c}\frac{dK}{K}\\ \frac{dU^*}{U^*}\end{array}\right]=-\frac{1}{\Delta}M^C\left[\begin{array}{c}\frac{d\tau}{\tau}\\ \frac{d\mu_u}{\mu_u+(1+c)A}\end{array}\right]$$

where Δ is the determinant of M:

$$\begin{aligned}\Delta&=\left[\left(1+\beta^{-1}D\right)E_K+C\frac{K}{\bar{K}-K}\right](E_U+\beta)-\left(E_K+\frac{\beta K}{\bar{K}-K}\right)\left[\left(1+\beta^{-1}D\right)E_U+C\right]\\&=(\beta+D-C)\left(E_K-\frac{K}{\bar{K}-K}E_U\right)=E_K-\frac{K}{\bar{K}-K}E_U\end{aligned}$$

and M^C is the co-factors matrix:

$$M^C=\left[\begin{array}{cc}E_U+\beta & -\left(1+\beta^{-1}D\right)E_U-C\\ -\left(E_K+\frac{\beta K}{\bar{K}-K}\right) & \left(1+\beta^{-1}D\right)E_K+C\frac{K}{\bar{K}-K}\end{array}\right]$$

Using the expressions for E_K and E_U, we get:

$$\Delta=E_K-\frac{K}{\bar{K}-K}E_U=\frac{(1-\beta+\beta\gamma)\,p\frac{Q_2}{K}-\gamma p\frac{Q_2^*}{\bar{K}-K}-(1-\gamma)\,A\,(1+c)\,\frac{U^*}{\bar{K}-K}}{\left(\frac{1-\beta+\beta\gamma}{\beta}Q_2+\gamma Q_2^*\right)pK^{-1}}$$

The denominator of Δ is straightforwardly positive. As for the numerator, using the equilibrium conditions, we get:

$$\begin{aligned}Num&=(1-\beta+\beta\gamma)\,p\frac{Q_2}{K}-\gamma p\frac{Q_2^*}{\bar{K}-K}-(1-\gamma)\,A\,(1+c)\,\frac{U^*}{\bar{K}-K}\\&=(1-\beta+\beta\gamma)\,p\frac{Q_2}{K}-\gamma p\frac{Q_2^*}{\bar{K}-K}-(1-\gamma)\,\frac{U^*}{\bar{K}-K}\left[(1-\beta)p\frac{Q_2^*}{U^*}-\mu_u\right]\\&=(1-\beta+\beta\gamma)\,p\frac{Q_2}{K}-\left[\gamma+(1-\gamma)\,(1-\beta)\right]p\frac{Q_2^*}{\bar{K}-K}+(1-\gamma)\,\frac{\mu_u U^*}{\bar{K}-K}\\&=(1-\beta+\beta\gamma)\,p\left(\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right)+(1-\gamma)\,\frac{\mu_u U^*}{\bar{K}-K}\\&=\frac{1-\beta+\beta\gamma}{\beta}\tau+\frac{(1-\gamma)\,K}{\bar{K}-K}\mu_u U^*>0\end{aligned}$$

so that $\Delta>0$.

Then, the diagonal terms of M^C being unambiguously positive and the two non diagonal terms unambiguously negative, we get the following signs for the impacts of an increase in τ and μ_u:

	$\frac{d\tau}{\tau}$	$\frac{d\mu_u}{\mu_u+(1+c)A}$
$\frac{dK}{K}$	-	+
$\frac{dU^*}{U^*}$	+	-

A.7 Impact of a risk premium and migration costs variation on the factors allocation (large transfer case with abundant capital endowment)

At equilibrium, we look at the impact of a risk premium and migration costs variations on the factors allocation among countries, when capital is worldly abundant and is invested in both countries ($K>0$ and $K^*>0$), case initially developed in section 3.4.

Let us remind the equilibrium conditions:

$$\tau = \beta Bp\left[\left(\frac{S}{K}\right)^{1-\beta} - \left(\frac{U^*}{\bar{K}-K}\right)^{1-\beta}\right] = \beta p\left[\frac{Q_2}{K} - \frac{Q_2^*}{\bar{K}-K}\right]$$

$$\mu_s = (1-\beta)\,Bp\left(\frac{K}{S}\right)^{\beta} - (1+c)\,A$$

$$\mu_u = (1-\beta)Bp\left(\frac{\bar{K}-K}{U^*}\right)^{\beta} - (1+c)\,A$$

where the world price of the industrial good is:

$$p = \frac{1-\gamma}{\gamma}\frac{Q_1+Q_1^*}{Q_2+Q_2^*} = \frac{1-\gamma}{\gamma}\frac{A\left[\bar{U}+c\bar{S}-(1+c)\,(U^*+S)\right]}{B\left[K^{\beta}S^{1-\beta}+\left(\bar{K}-K\right)^{\beta}(U^*)^{1-\beta}\right]}$$

The logarithmic derivative of the price is:

$$\begin{aligned}
\frac{dp}{p} &= \frac{d\,(Q_1+Q_1^*)}{Q_1+Q_1^*} - \frac{d\,(Q_2+Q_2^*)}{Q_2+Q_2^*}\\
&= -\frac{A\,(1+c)\,(dS+dU^*)}{Q_1+Q_1^*} - \frac{Q_2\left(\beta\frac{dK}{K}+(1-\beta)\,\frac{dS}{S}\right)+Q_2^*\left((1-\beta)\,\frac{dU^*}{U^*}-\beta\frac{dK}{\bar{K}-K}\right)}{Q_2+Q_2^*}\\
&= -\frac{A\,(1+c)}{Q_1+Q_1^*}\,(dS+dU^*) - \frac{\beta Q_2}{Q_2+Q_2^*}\frac{dK}{K} - \frac{(1-\beta)\,Q_2}{Q_2+Q_2^*}\frac{dS}{S} - \frac{(1-\beta)\,Q_2^*}{Q_2+Q_2^*}\frac{dU^*}{U^*} + \frac{\beta Q_2^*}{Q_2+Q_2^*}\frac{dK}{\bar{K}-K}\\
&= -\frac{A\,(1+c)}{Q_1+Q_1^*}\,(dS+dU^*) + \frac{\beta}{Q_2+Q_2^*}\left(\frac{KQ_2^*}{\bar{K}-K}-Q_2\right)\frac{dK}{K} - \frac{(1-\beta)\,Q_2}{Q_2+Q_2^*}\frac{dS}{S} - \frac{(1-\beta)\,Q_2^*}{Q_2+Q_2^*}\frac{dU^*}{U^*}\\
&= B_K\frac{dK}{K} - B_S\frac{dS}{S} - B_U\frac{dU^*}{U^*}
\end{aligned}$$

with:

$$B_K = -\frac{\beta K}{Q_2+Q_2^*}\left[\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}\right] = -\frac{\tau K/p}{Q_2+Q_2^*} < 0$$

$$B_S = \frac{A\,(1+c)}{Q_1+Q_1^*}S + \frac{(1-\beta)\,Q_2}{Q_2+Q_2^*} > 0$$

$$B_U = \frac{A\,(1+c)}{Q_1+Q_1^*}U^* + \frac{(1-\beta)\,Q_2^*}{Q_2+Q_2^*} > 0$$

and:

$$\begin{aligned}
B_U + B_S &= \frac{A\,(1+c)\,(S+U^*)}{Q_1+Q_1^*} + 1 - \beta\\
&= \frac{1}{\frac{\bar{U}+c\bar{S}}{(1+c)(U^*+S)}-1} + 1 - \beta\\
&= \frac{1}{1-(1+c)\,\frac{S+U^*}{\bar{U}+c\bar{S}}} - \beta
\end{aligned}$$

The logarithmic derivatives of the equilibrium conditions are:

$$\begin{aligned}
\frac{d\tau}{\tau} &= \frac{dp}{p} + (1-\beta)\,\frac{\frac{Q_2}{K}\left(\frac{dS}{S}-\frac{dK}{K}\right)-\frac{Q_2^*}{\bar{K}-K}\left(\frac{dU^*}{U^*}+\frac{dK}{\bar{K}-K}\right)}{\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}} \\
&= \frac{dp}{p} - A_K\frac{dK}{K} + A_S\frac{dS}{S} - A_U\frac{dU^*}{U^*} \\
\frac{d\mu_s}{\mu_s+(1+c)\,A} &= \frac{dp}{p} + \beta\left(\frac{dK}{K}-\frac{dS}{S}\right) \\
\frac{d\mu_u}{\mu_u+(1+c)\,A} &= \frac{dp}{p} - \beta\left(\frac{dK}{\bar{K}-K}+\frac{dU^*}{U^*}\right)
\end{aligned}$$

where:

$$\begin{aligned}
A_K &= (1-\beta)\,\frac{\frac{Q_2}{K}+\frac{K}{\bar{K}-K}\frac{Q_2^*}{\bar{K}-K}}{\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}} = A_S + \frac{K}{\bar{K}-K}A_U > 0 \\
A_S &= \frac{(1-\beta)\,\frac{Q_2}{K}}{\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}} = (1-\beta)\,\frac{\beta p}{\tau}\frac{Q_2}{K} > 0 \\
A_U &= \frac{(1-\beta)\,\frac{Q_2^*}{\bar{K}-K}}{\frac{Q_2}{K}-\frac{Q_2^*}{\bar{K}-K}} = (1-\beta)\,\frac{\beta p}{\tau}\frac{Q_2^*}{\bar{K}-K} > 0
\end{aligned}$$

Combining the last two equilibrium conditions, we find:

$$\frac{dK}{K} = \frac{\bar{K}-K}{\bar{K}}\left(\frac{dS}{S}-\frac{dU^*}{U^*}\right) + \frac{\bar{K}-K}{\beta\bar{K}}\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)$$

Then:

$$\begin{aligned}
\frac{d\tau}{\tau} =& \frac{dp}{p} - \frac{\bar{K}-K}{\beta\bar{K}}A_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right) \\
&+ \left(A_S-\frac{\bar{K}-K}{\bar{K}}A_K\right)\frac{dS}{S} + \left(\frac{\bar{K}-K}{\bar{K}}A_K-A_U\right)\frac{dU^*}{U^*} \\
\frac{d\mu_s}{\mu_s+(1+c)\,A} =& \frac{dp}{p} - \beta\left(\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}\right) + \frac{\bar{K}-K}{\bar{K}}\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right) \\
\frac{d\mu_u}{\mu_u+(1+c)\,A} =& \frac{dp}{p} - \beta\left(\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}\right) - \frac{K}{\bar{K}}\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)
\end{aligned}$$

where noting that $A_S - A_U = 1-\beta$:

$$\begin{aligned}
A_S - \frac{\bar{K}-K}{\bar{K}}A_K &= A_S - \frac{\bar{K}-K}{\bar{K}}\left(A_S+\frac{K}{\bar{K}-K}A_U\right) = \frac{K}{\bar{K}}\,(A_S-A_U) = (1-\beta)\,\frac{K}{\bar{K}} \\
\frac{\bar{K}-K}{\bar{K}}A_K - A_U &= \frac{\bar{K}-K}{\bar{K}}\left(A_S+\frac{K}{\bar{K}-K}A_U\right) - A_U = \frac{\bar{K}-K}{\bar{K}}\,(A_S-A_U) = (1-\beta)\,\frac{\bar{K}-K}{\bar{K}}
\end{aligned}$$

Then, by reducing the last two equations to the same final equation, we find:

$$\begin{aligned}
\frac{d\tau}{\tau} =& \frac{dp}{p} + (1-\beta)\,\frac{K}{\bar{K}}\frac{dS}{S} + (1-\beta)\,\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*} \\
&- \frac{\bar{K}-K}{\beta\bar{K}}A_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right) \\
\beta\left(\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}\right) =& \frac{dp}{p} - \frac{K}{\bar{K}}\frac{d\mu_s}{\mu_s+(1+c)\,A} - \frac{\bar{K}-K}{\bar{K}}\frac{d\mu_u}{\mu_u+(1+c)\,A}
\end{aligned}$$

or equivalently:

$$(1-\beta)\left(\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}\right)=\frac{d\tau}{\tau}-\frac{dp}{p}+\frac{\bar{K}-K}{\beta\bar{K}}A_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)$$

$$\beta\left(\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}\right)=\frac{dp}{p}-\frac{K}{\bar{K}}\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{\bar{K}-K}{\bar{K}}\frac{d\mu_u}{\mu_u+(1+c)\,A}$$

so that:

$$\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}=\frac{d\tau}{\tau}+\left(\frac{\bar{K}-K}{\beta\bar{K}}A_K-\frac{K}{\bar{K}}\right)\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{\bar{K}-K}{\beta\bar{K}}\left(A_K+\beta\right)\frac{d\mu_u}{\mu_u+(1+c)\,A}$$

$$\begin{aligned}\frac{dp}{p}&=\beta\frac{d\tau}{\tau}+\left(\frac{\bar{K}-K}{\bar{K}}A_K+(1-\beta)\frac{K}{\bar{K}}\right)\frac{d\mu_s}{\mu_s+(1+c)\,A}+(1-\beta-A_K)\frac{\bar{K}-K}{\bar{K}}\frac{d\mu_u}{\mu_u+(1+c)\,A}\\&=\beta\frac{d\tau}{\tau}+A_S\frac{d\mu_s}{\mu_s+(1+c)\,A}-A_U\frac{d\mu_u}{\mu_u+(1+c)\,A}\end{aligned}$$

Moreover:

$$\begin{aligned}\frac{dp}{p}=&B_K\frac{dK}{K}-B_S\frac{dS}{S}-B_U\frac{dU^*}{U^*}\\=&B_K\left[\frac{\bar{K}-K}{\bar{K}}\left(\frac{dS}{S}-\frac{dU^*}{U^*}\right)+\frac{\bar{K}-K}{\beta\bar{K}}\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)\right]\\&-B_S\frac{dS}{S}-B_U\frac{dU^*}{U^*}\\=&\frac{\bar{K}-K}{\beta\bar{K}}B_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)+\left(\frac{\bar{K}-K}{\bar{K}}B_K-B_S\right)\frac{dS}{S}\\&-\left(\frac{\bar{K}-K}{\bar{K}}B_K+B_U\right)\frac{dU^*}{U^*}\end{aligned}$$

and then:

$$\begin{aligned}\left(\frac{\bar{K}-K}{\bar{K}}B_K+B_U\right)\frac{dU^*}{U^*}-\left(\frac{\bar{K}-K}{\bar{K}}B_K-B_S\right)\frac{dS}{S}=&\frac{\bar{K}-K}{\beta\bar{K}}B_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)-\frac{dp}{p}\\=&\frac{\bar{K}-K}{\beta\bar{K}}B_K\left(\frac{d\mu_s}{\mu_s+(1+c)\,A}-\frac{d\mu_u}{\mu_u+(1+c)\,A}\right)\\&-\beta\frac{d\tau}{\tau}-A_S\frac{d\mu_s}{\mu_s+(1+c)\,A}+A_U\frac{d\mu_u}{\mu_u+(1+c)\,A}\\=&-\beta\frac{d\tau}{\tau}+\left(\frac{\bar{K}-K}{\beta\bar{K}}B_K-A_S\right)\frac{d\mu_s}{\mu_s+(1+c)\,A}\\&+\left(A_U-\frac{\bar{K}-K}{\beta\bar{K}}B_K\right)\frac{d\mu_u}{\mu_u+(1+c)\,A}\end{aligned}$$

so that:

$$\begin{aligned}\frac{K}{\bar{K}}\frac{dS}{S}+\frac{\bar{K}-K}{\bar{K}}\frac{dU^*}{U^*}=&\frac{d\tau}{\tau}+\left(\frac{\bar{K}-K}{\beta\bar{K}}A_K-\frac{K}{\bar{K}}\right)\frac{d\mu_s}{\mu_s+(1+c)\,A}\\&-\frac{\bar{K}-K}{\beta\bar{K}}\left(A_K+\beta\right)\frac{d\mu_u}{\mu_u+(1+c)\,A}\\\left(B_S-\frac{\bar{K}-K}{\bar{K}}B_K\right)\frac{dS}{S}+\left(\frac{\bar{K}-K}{\bar{K}}B_K+B_U\right)\frac{dU^*}{U^*}=&-\beta\frac{d\tau}{\tau}+\left(\frac{\bar{K}-K}{\beta\bar{K}}B_K-A_S\right)\frac{d\mu_s}{\mu_s+(1+c)\,A}\\&+\left(A_U-\frac{\bar{K}-K}{\beta\bar{K}}B_K\right)\frac{d\mu_u}{\mu_u+(1+c)\,A}\end{aligned}$$

or equivalently:

$$\Omega \begin{bmatrix} \frac{dS}{S} \\ \frac{dU^*}{U^*} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\bar{K}-K}{\beta \bar{K}} A_K - \frac{K}{\bar{K}} & -\frac{\bar{K}-K}{\beta \bar{K}} (A_K + \beta) \\ -\beta & \frac{\bar{K}-K}{\beta \bar{K}} B_K - A_S & A_U - \frac{\bar{K}-K}{\beta \bar{K}} B_K \end{bmatrix} \begin{bmatrix} \frac{d\tau}{\tau} \\ \frac{d\mu_s}{\mu_s+(1+c)A} \\ \frac{d\mu_u}{\mu_u+(1+c)A} \end{bmatrix}$$

where:

$$\Omega = \begin{bmatrix} \frac{K}{\bar{K}} & \frac{\bar{K}-K}{\bar{K}} \\ B_S - \frac{\bar{K}-K}{\bar{K}} B_K & \frac{\bar{K}-K}{\bar{K}} B_K + B_U \end{bmatrix}$$

The determinant of Ω is

$$\begin{aligned} \Delta &= \frac{K}{\bar{K}} \left(\frac{\bar{K}-K}{\bar{K}} B_K + B_U \right) - \frac{\bar{K}-K}{\bar{K}} \left[B_S - \frac{\bar{K}-K}{\bar{K}} B_K \right] \\ &= \frac{\bar{K}-K}{\bar{K}} (B_K - B_S) + \frac{K}{\bar{K}} B_U \end{aligned}$$

Then:

$$\begin{bmatrix} \frac{dS}{S} \\ \frac{dU^*}{U^*} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{\bar{K}-K}{\bar{K}} B_K + B_U & -\frac{\bar{K}-K}{\bar{K}} \\ \frac{\bar{K}-K}{\bar{K}} B_K - B_S & \frac{K}{\bar{K}} \end{bmatrix} \begin{bmatrix} 1 & \frac{\bar{K}-K}{\beta \bar{K}} A_K - \frac{K}{\bar{K}} & -\frac{\bar{K}-K}{\beta \bar{K}} (A_K + \beta) \\ -\beta & \frac{\bar{K}-K}{\beta \bar{K}} B_K - A_S & A_U - \frac{\bar{K}-K}{\beta \bar{K}} B_K \end{bmatrix} \begin{bmatrix} \frac{d\tau}{\tau} \\ \frac{d\mu_s}{\mu_s+(1+c)A} \\ \frac{d\mu_u}{\mu_u+(1+c)A} \end{bmatrix}$$

and, after calculations:

$$\begin{aligned} \frac{dS}{S} &= \frac{M_{S\tau}}{\Delta} \frac{d\tau}{\tau} + \frac{M_{SS}}{\Delta} \frac{d\mu_s}{\mu_s + (1+c) A} + \frac{M_{SU}}{\Delta} \frac{d\mu_u}{\mu_u + (1+c) A} \\ \frac{dU^*}{U^*} &= \frac{M_{U\tau}}{\Delta} \frac{d\tau}{\tau} + \frac{M_{US}}{\Delta} \frac{d\mu_s}{\mu_s + (1+c) A} + \frac{M_{UU}}{\Delta} \frac{d\mu_u}{\mu_u + (1+c) A} \end{aligned}$$

where, using the equalities $A_K = A_S + \frac{K}{\bar{K}-K} A_U$ and $A_S - A_U = 1 - \beta$:

$$\begin{aligned} M_{S\tau} &= \frac{\bar{K}-K}{\bar{K}} B_K + B_U + \beta \frac{\bar{K}-K}{\bar{K}} = \frac{\bar{K}-K}{\bar{K}} (B_K + \beta) + B_U \\ M_{SS} &= \left(\frac{\bar{K}-K}{\bar{K}} B_K + B_U \right) \left(\frac{\bar{K}-K}{\beta \bar{K}} A_K - \frac{K}{\bar{K}} \right) - \frac{\bar{K}-K}{\bar{K}} \left(\frac{\bar{K}-K}{\beta \bar{K}} B_K - A_S \right) \\ &= \frac{\bar{K}-K}{\beta \bar{K}} \left[\left(\frac{\bar{K}-K}{\bar{K}} A_K + (1-\beta) \frac{K}{\bar{K}} - 1 \right) B_K + \beta A_S + \left(A_K - \frac{\beta K}{\bar{K}-K} \right) B_U \right] \\ &= \frac{\bar{K}-K}{\beta \bar{K}} \left[(A_S - 1) B_K + \beta A_S + \left(A_K - \frac{\beta K}{\bar{K}-K} \right) B_U \right] \\ M_{SU} &= -\frac{\bar{K}-K}{\beta \bar{K}} (A_K + \beta) \left(\frac{\bar{K}-K}{\bar{K}} B_K + B_U \right) - \frac{\bar{K}-K}{\bar{K}} \left(A_U - \frac{\bar{K}-K}{\beta \bar{K}} B_K \right) \\ &= -\frac{\bar{K}-K}{\beta \bar{K}} \left[(A_K + \beta - 1) \frac{\bar{K}-K}{\bar{K}} B_K + (A_K + \beta) B_U + \beta A_U \right] \\ &= -\frac{\bar{K}-K}{\beta \bar{K}} [(B_K + \beta) A_U + (A_K + \beta) B_U] \end{aligned}$$

and:

$$
\begin{aligned}
M_{U\tau} &= \frac{\bar{K}-K}{\bar{K}} B_K - B_S - \beta \frac{K}{\bar{K}} \\
M_{US} &= \left(\frac{\bar{K}-K}{\bar{K}} B_K - B_S \right) \left(\frac{\bar{K}-K}{\beta \bar{K}} A_K - \frac{K}{\bar{K}} \right) + \frac{K}{\bar{K}} \left(\frac{\bar{K}-K}{\beta \bar{K}} B_K - A_S \right) \\
&= \frac{\bar{K}-K}{\beta \bar{K}} \left[\left(\frac{\bar{K}-K}{\bar{K}} B_K - B_S \right) \left(A_K - \frac{\beta K}{\bar{K}-K} \right) + \frac{K}{\bar{K}} \left(B_K - \frac{\beta \bar{K}}{\bar{K}-K} A_S \right) \right] \\
&= \frac{\bar{K}-K}{\beta \bar{K}} \left[A_S B_K - A_K B_S + \frac{\beta K}{\bar{K}-K} (B_S - A_S) \right] \\
M_{UU} &= -\frac{\bar{K}-K}{\beta \bar{K}} (A_K + \beta) \left(\frac{\bar{K}-K}{\bar{K}} B_K - B_S \right) + \frac{K}{\bar{K}} \left(A_U - \frac{\bar{K}-K}{\beta \bar{K}} B_K \right) \\
&= \frac{K}{\bar{K}} A_U - \frac{\bar{K}-K}{\beta \bar{K}} \left[\left((A_K + \beta) \frac{\bar{K}-K}{\bar{K}} + \frac{K}{\bar{K}} \right) B_K - (A_K + \beta) B_S \right] \\
&= \frac{\bar{K}-K}{\beta \bar{K}} \left[(A_K + \beta) B_S - (A_U + 1) B_K + \frac{\beta K}{\bar{K}-K} A_U \right]
\end{aligned}
$$

Combining the two expressions, we get:

$$
\frac{dS}{S} - \frac{dU^*}{U^*} = \frac{D_\tau}{\Delta} \frac{d\tau}{\tau} + \frac{D_S}{\Delta} \frac{d\mu_s}{\mu_s + (1+c) A} + \frac{D_U}{\Delta} \frac{d\mu_u}{\mu_u + (1+c) A}
$$

where:

$$
\begin{aligned}
D_\tau &= M_{S\tau} - M_{U\tau} = B_U + B_S + \beta \\
D_S &= M_{SS} - M_{US} = \left(\frac{\bar{K}-K}{\beta \bar{K}} A_K - \frac{K}{\bar{K}} \right) (B_U + B_S) - \frac{\bar{K}-K}{\beta \bar{K}} B_K + A_S \\
D_U &= M_{SU} - M_{UU} = -\frac{\bar{K}-K}{\beta \bar{K}} \left[(A_K + \beta)(B_U + B_S) - B_K \right] - A_U
\end{aligned}
$$

hence:

$$
\begin{aligned}
\frac{dK}{K} =& \frac{\bar{K}-K}{\bar{K}} \left(\frac{dS}{S} - \frac{dU^*}{U^*} \right) + \frac{\bar{K}-K}{\beta \bar{K}} \left(\frac{d\mu_s}{\mu_s + (1+c) A} - \frac{d\mu_u}{\mu_u + (1+c) A} \right) \\
=& \frac{\bar{K}-K}{\bar{K}} \left(\frac{D_\tau}{\Delta} \frac{d\tau}{\tau} + \frac{D_S}{\Delta} \frac{d\mu_s}{\mu_s + (1+c) A} + \frac{D_U}{\Delta} \frac{d\mu_u}{\mu_u + (1+c) A} \right) \\
&+ \frac{\bar{K}-K}{\beta \bar{K}} \left(\frac{d\mu_s}{\mu_s + (1+c) A} - \frac{d\mu_u}{\mu_u + (1+c) A} \right) \\
=& \frac{\bar{K}-K}{\bar{K}\Delta} D_\tau \frac{d\tau}{\tau} + \frac{\bar{K}-K}{\beta \bar{K}} \left(1 + \frac{\beta D_S}{\Delta} \right) \frac{d\mu_s}{\mu_s + (1+c) A} + \frac{\bar{K}-K}{\beta \bar{K}} \left(\frac{\beta D_U}{\Delta} - 1 \right) \frac{d\mu_u}{\mu_u + (1+c) A}
\end{aligned}
$$

Knowing that $A_K = A_S + \frac{K}{\bar{K}-K} A_U$, $A_S - A_U = 1 - \beta$, $A_K > 0$, $A_S > 0$, $A_U > 0$, $B_K < 0$, $B_S > 0$, $B_U > 0$, and $\frac{U^*}{\bar{K}-K} < \frac{S}{K}$, we find the following signs:

$$
\begin{aligned}
\Delta = & \frac{\bar{K}-K}{\bar{K}} \left(B_K - B_S\right) + \frac{K}{\bar{K}} B_U \\
= & \frac{\bar{K}-K}{\bar{K}} \left(-\frac{\tau K/p}{Q_2 + Q_2^*} - \frac{A\left(1+c\right)}{Q_1 + Q_1^*} S - \frac{\left(1-\beta\right) Q_2}{Q_2 + Q_2^*} \right) \\
& + \frac{K}{\bar{K}} \left(\frac{A\left(1+c\right)}{Q_1 + Q_1^*} U^* + \frac{\left(1-\beta\right) Q_2^*}{Q_2 + Q_2^*} \right) \\
= & - \frac{K\left(\bar{K}-K\right)}{\bar{K}} \left[\frac{\tau/p}{Q_2 + Q_2^*} + \frac{A\left(1+c\right)}{Q_1 + Q_1^*} \left(\frac{S}{K} - \frac{U^*}{\bar{K}-K} \right) + \frac{1-\beta}{Q_2 + Q_2^*} \left(\frac{Q_2}{K} - \frac{Q_2^*}{\bar{K}-K} \right) \right] < 0
\end{aligned}
$$

and:

$$
B_K + \beta = \frac{\bar{K}}{\bar{K}-K} \frac{\beta Q_2^*}{Q_2 + Q_2^*} > 0
$$

so that:

$$
\begin{aligned}
M_{S\tau} &= \frac{\bar{K}-K}{\bar{K}} \left(B_K + \beta\right) + B_U = \frac{\beta Q_2^*}{Q_2 + Q_2^*} + B_U > 0 \\
M_{U\tau} &= \frac{\bar{K}-K}{\bar{K}} B_K - B_S - \beta \frac{K}{\bar{K}} < 0 \\
M_{SU} &= -\frac{\bar{K}-K}{\beta \bar{K}} \left[\left(B_K + \beta\right) A_U + \left(A_K + \beta\right) B_U \right] < 0 \\
M_{UU} &= \frac{\bar{K}-K}{\beta \bar{K}} \left[\left(A_K + \beta\right) B_S - \left(A_U + 1\right) B_K + \frac{\beta K}{\bar{K}-K} A_U \right] > 0 \\
D_\tau &= B_U + B_S + \beta > 0
\end{aligned}
$$

and:

$$
\begin{aligned}
\beta D_U - \Delta &= \frac{\bar{K}-K}{\bar{K}} \left[B_K - \left(A_K + \beta\right)\left(B_U + B_S\right) \right] - \beta A_U - \frac{\bar{K}-K}{\bar{K}} \left(B_K - B_S\right) - \frac{K}{\bar{K}} B_U \\
&= \frac{\bar{K}-K}{\bar{K}} B_S - \frac{K}{\bar{K}} B_U - \frac{\bar{K}-K}{\bar{K}} \left(A_K + \beta\right)\left(B_U + B_S\right) - \beta A_U \\
&= \frac{\bar{K}-K}{\bar{K}} \left(B_S + B_U\right) - B_U - \frac{\bar{K}-K}{\bar{K}} \left(A_K + \beta\right)\left(B_U + B_S\right) - \beta A_U \\
&= -\frac{\bar{K}-K}{\bar{K}} \left(A_K + \beta - 1\right)\left(B_U + B_S\right) - \beta A_U - B_U \\
&= -A_U \left(B_U + B_S + \beta\right) - B_U < 0
\end{aligned}
$$

The signs of the following coefficients have not been determined yet:

$$
\begin{aligned}
M_{SS} &= \frac{\bar{K}-K}{\beta\bar{K}}\left[\left(A_S-1\right)B_K+\beta A_S+\left(A_K-\frac{\beta K}{\bar{K}-K}\right)B_U\right] \\
M_{US} &= \frac{\bar{K}-K}{\beta\bar{K}}\left[A_S B_K - A_K B_S + \frac{\beta K}{\bar{K}-K}\left(B_S-A_S\right)\right] \\
\beta D_S+\Delta &= \left(\frac{\bar{K}-K}{\bar{K}}A_K-\beta\frac{K}{\bar{K}}\right)\left(B_U+B_S\right)-\frac{\bar{K}-K}{\bar{K}}B_S+\frac{K}{\bar{K}}B_U+\beta A_S \\
&= \left[\frac{\bar{K}-K}{\bar{K}}A_K+\left(1-\beta\right)\frac{K}{\bar{K}}\right]B_U+\left(\frac{\bar{K}-K}{\bar{K}}A_K-\beta\frac{K}{\bar{K}}-\frac{\bar{K}-K}{\bar{K}}\right)B_S+\beta A_S \\
&= \left[\frac{\bar{K}-K}{\bar{K}}A_K+\left(1-\beta\right)\frac{K}{\bar{K}}\right]B_U+\left[\frac{\bar{K}-K}{\bar{K}}A_K+\left(1-\beta\right)\frac{K}{\bar{K}}-1\right]B_S+\beta A_S \\
&= \left[\frac{\bar{K}-K}{\bar{K}}A_K+\left(1-\beta\right)\frac{K}{\bar{K}}\right]B_U+\left(A_S-1\right)B_S+\beta A_S
\end{aligned}
$$

To summarize, we get the following results:

$$
\begin{aligned}
\frac{dS}{S} &= \frac{M_{S\tau}}{\Delta}\frac{d\tau}{\tau}+\frac{M_{SS}}{\Delta}\frac{d\mu_s}{\mu_s+\left(1+c\right)A}+\frac{M_{SU}}{\Delta}\frac{d\mu_u}{\mu_u+\left(1+c\right)A} \\
\frac{dU^*}{U^*} &= \frac{M_{U\tau}}{\Delta}\frac{d\tau}{\tau}+\frac{M_{US}}{\Delta}\frac{d\mu_s}{\mu_s+\left(1+c\right)A}+\frac{M_{UU}}{\Delta}\frac{d\mu_u}{\mu_u+\left(1+c\right)A} \\
\frac{dK}{K} &= \frac{\bar{K}-K}{\bar{K}\Delta}D_\tau\frac{d\tau}{\tau}+\frac{\bar{K}-K}{\beta\bar{K}}\left(1+\frac{\beta D_S}{\Delta}\right)\frac{d\mu_s}{\mu_s+\left(1+c\right)A}+\frac{\bar{K}-K}{\beta\bar{K}}\left(\frac{\beta D_U}{\Delta}-1\right)\frac{d\mu_u}{\mu_u+\left(1+c\right)A}
\end{aligned}
$$

with $\Delta<0$, $M_{S\tau}>0$, $M_{U\tau}<0$, $M_{SU}>0$, $M_{UU}>0$, $D_\tau>0$, $\beta D_U-\Delta<0$, so that:

	$\frac{d\tau}{\tau}$	$\frac{d\mu_u}{\mu_u+(1+c)A}$
$\frac{dK}{K}$	-	+
$\frac{dS}{S}$	-	+
$\frac{dU^*}{U^*}$	+	-